ON THE PRINCIPLE OF COMPLEMENTARY VIRTUAL WORK IN THE NON-LINEAR THEORY OF ELASTICITY

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Abstract. A generalization of the principle of complementary virtual work in the current configuration is proposed by introducing the orthogonal rotation tensor as an independent variable. Applicability of the modified principle to (numerical) analysis of non-linear elasticity problems assumes the existence of an invertible constitutive relation between the Cauchy stress tensor and the left stretch tensor. It is shown that when transforming it into the reference configuration, the generalized principle of complementary virtual work is equivalent to Fraeis de Veubeke’s two-field dual-mixed principle, provided an appropriate constitutive equation between the Biot stress tensor and the right stretch tensor is taken into account. The independent equation system of non-linear elasticity in terms of stresses and rotations is derived, both in the current and reference configurations, using six independent first-order stress functions.

Keywords: Principle of complementary virtual work, non-linear elasticity, rotation tensor

1. Introduction

Applicability of the classical principle of complementary virtual work in non-linear elasticity is limited because (i) in the current configuration the Cauchy stress tensor and the displacement gradient tensor appearing in the principle are not work-conjugate stress and strain measures, thus they cannot be related to each other by constitutive equations; (ii) in the reference configuration the principle contains the first Piola-Kirchhoff stress tensor and the displacement gradient tensor which are work conjugate stress and strain measures, the constitutive relations between them, however, cannot be inverted uniquely. Due to the above mentioned restrictions, the principle of complementary virtual work has primarily theoretical importance in the non-linear theory of elasticity and is usually written in the reference configuration, see for example Novozhilov [1], Zubov [2], Lur’e [3], Washizu [4].

The complementary virtual work theorem is closely related to the principle of stationary complementary energy, as the latter can be derived from the former when invertible constitutive equations can be taken into account. The problem of construction complementary variational principles in non-linear elasticity in terms of stresses alone has been investigated by several authors. The key issue is to find appropriate work-conjugate stress and strain measures for which the constitutive equation is uniquely invertible, i.e. the strain tensor can be expressed as a function of the corresponding conjugate stress tensor. If such a work-conjugate stress and strain measure
exists, a complementary strain energy density can be derived from a given strain energy density through a Legendre transformation.

Levinson [5] pointed out that no complementary energy density as a function of the second Piola-Kirchhoff stress tensor alone can be derived. Construction of complementary energy density as a function of the first Piola-Kirchhoff stress tensor as well as the invertibility of the constitutive equations between the first Piola-Kirchhoff stress tensor and the deformation/displacement gradient tensor has been investigated, among others, by Novozhilov [1], Truesdell–Noll [6], Zubov [2], Fraeijs de Veubeke [7], Christoffersen [8], Koiter [9], Dill [10], Ogden [11]. The final conclusion was that it is impossible, in general, to derive a unique complementary strain energy density in terms of the first Piola-Kirchhoff stresses alone, since the inversion of the stress-strain relation between the first Piola-Kirchhoff stress tensor and the deformation/displacement gradient tensor is not unique [11]. It should be noted, however, that for some special cases, when the sign problem appearing in the inversion of the stress-strain relation is determinable by practical considerations, the principle of stationary complementary energy is applicable to solving non-linear elasticity problems [2][9][13].

The right and perhaps the best way to define a complementary variational principle in non-linear elasticity is using the Biot stress tensor and the right stretch tensor as work-conjugate (Lagrangian) stress and strain measures. In this case the stress-strain relations are uniquely invertible and the complementary strain energy density can always be expressed in terms of the Biot stress tensor alone. Recognizing these facts, a complementary energy-based dual-mixed principle in terms of the first Piola-Kirchhoff stress tensor, depending on the Biot stress tensor and the orthogonal rotation tensor, was derived by Fraeijs de Veubeke [7] in the reference configuration. The possible relationship between this two-field principle and a generalized complementary virtual work theorem, which should be independent of the actual constitutive equations, was not investigated in [7], however. Either, no attempt was made to transform the derived complementary energy principle into the current configuration (which is, otherwise, a rather complicated task), or to derive an applicable principle of complementary virtual work in the current configuration, independently of the developments and results presented in [7].

Considering the above mentioned facts and limitations regarding the applicability of the principle of complementary virtual work theorem in non-linear elasticity, this paper deals with a possible generalization of the classical principle by taking both the stress tensor and the orthogonal rotation tensor as independent variables. Section 2 focuses on the generalization of the classical principle in the current configuration. The orthogonal rotation tensor is introduced into the principle by considering the Cauchy stress tensor to be not a priori symmetric. The rotational equilibrium for the Cauchy stresses becomes a variational result. The modified principle is applicable to numerical analysis of non-linear elasticity problems when invertible constitutive relation between the Cauchy stress tensor and the left stretch tensor exists. Independent equation system of nonlinear elasticity in terms of the Cauchy stresses and orthogonal rotations are derived from the generalized principle in the current configuration, using six independent first-order stress functions.
In Section 3, the generalized principle is transformed into the reference configuration. It is pointed out that, provided an appropriate constitutive equation between the symmetrized Biot stress tensor and the right stretch tensor is taken into account, the generalized principle is equivalent to Fraeijs de Veubeke’s two-field dual-mixed principle. Independent equation system of non-linear elasticity in terms of the first Piola-Kirchhoff stresses and orthogonal rotations is derived, using six independent first-order stress functions. Differences between the present and a former (incomplete) derivation given in [7] are emphasized.

**Notation.** Let the initial or reference configuration of an elastic body (at time \( t = 0 \)) be denoted by \( {}^{0}\Omega \) and its deformed or current configuration at time \( t > 0 \) by \( {}^{'}\Omega \). Points in the reference and current configurations will be denoted by \( \mathbf{X} \) and \( \mathbf{x} \), respectively. It is assumed that \( {}^{0}\Omega \) as well as \( {}^{'}\Omega \) are simply-connected and bounded by the sufficiently smooth boundaries \( \partial \Gamma = \partial \Gamma_{\sigma} \cup \partial \Gamma_{u} \) (\( \partial \Gamma_{\sigma} \cap \partial \Gamma_{u} = 0 \)) and \( \partial \Gamma' = \partial \Gamma_{\sigma}' \cup \partial \Gamma_{u}' \) (\( \partial \Gamma_{\sigma}' \cap \partial \Gamma_{u}' = 0 \)) and that these boundary parts with outward unit normals \( \mathbf{n} \) and \( \mathbf{n}' \) are separated, according to Figure 1, by the boundary curves \( \gamma' \) and \( \gamma \), respectively. The mass densities at the reference and current configurations are denoted by \( \rho \) and \( \rho' \). The elastic body is subjected to body forces of density \( \rho \mathbf{b} \) in \( {}^{0}\Omega \) \( \mathbf{b} \) is the body force density per unit mass), a surface load of density \( \mathbf{p} \) on \( \partial \Gamma_{\sigma} \), whereas on the boundary part \( \partial \Gamma_{u} \), a displacement field denoted by \( \mathbf{u} \) is prescribed.

![Figure 1. Elastic body: Reference and current configurations](image)

Throughout this paper, invariant notation of tensors and tensor operations will be used. Scalar, vectorial, tensorial and inner product of two tensors will be denoted, respectively, by \( \cdot, \times, \otimes \) and \( : \). Differential operations divergence, curl and gradient on tensor variables will be denoted by div, curl and grad with respect to the metric of the current configuration and by Div, Curl, and Grad with respect to the metric of the reference configuration. For the divergence, curl and gradient of the arbitrary, differentiable second-order tensor \( \mathbf{S} \), definitions of Gurtin [14][15] are employed, i.e.

\[
\text{div} \mathbf{S} = \nabla \cdot \mathbf{S}^T, \quad (1.1)
\]
\[
\text{curl} \mathbf{S} = \nabla \times \mathbf{S}^T, \quad (1.2)
\]
\[
\text{grad} \mathbf{S} = \nabla \otimes \mathbf{S}^T, \quad (1.3)
\]
where a $\Gamma$ in the superscript stands for the transpose. It is also convenient to introduce the surface divergence operator $\text{div}_\Gamma$ acting on the surface $\Gamma$ as

$$\text{div}_\Gamma S := (n \times \nabla) \cdot S^T,$$

(1.4)

where $n$ is the outward (unit) normal to $\Gamma$. This operator involves only tangential derivatives of $S$ on $\Gamma$.

2. The principle of complementary virtual work and its generalization in the current configuration

2.1. The classical principle. The classical complementary virtual work theorem states that if equality

$$\int_{\Omega} (1 - F^{-1}) : \delta \tau \ d'\Omega = \int_{\Gamma_a} \bar{u} \cdot \delta \tau \cdot n \ d'\Gamma$$

(2.1)

holds in the current configuration for all statically admissible Cauchy stresses $\tau$, i.e. if its variation $\delta \tau$ satisfies the translational equilibrium equations

$$\text{div} \delta \tau = 0 \quad x \in '\Omega$$

(2.2)

and the stress boundary conditions

$$\delta \tau \cdot n = 0 \quad x \in '\Gamma_a,$$

(2.3)

and the symmetry condition for the Cauchy stress tensor

$$\tau - \tau^T = 0 \quad x \in '\Omega$$

(2.4)

is also assumed to be a priori satisfied, then the tensor $D := 1 - F^{-1}$ is the displacement gradient tensor and $F^{-1}$ is the inverse deformation gradient tensor (the unit tensor is denoted by $1$). It can also be pointed out that when (2.1) holds, the Riemann-Christoffel curvature tensor of the current metric vanishes, which is a compatibility condition for the metric tensor of the deformed configuration, see Kozák [16].

Applicability of (2.1) is restricted, however, by the fact that the Cauchy stress tensor and the displacement gradient tensor (or the inverse deformation gradient) are not work-conjugate stress and strain measures and, unless the deformation is rotation-free, they cannot be related to each other by constitutive equations. In other words, neither $D$ nor $F^{-1}$ can be expressed as a function of $\tau$, which means that the principle of complementary virtual work in its classical form is not suitable for numerical analysis of non-linear elasticity problems. In addition, as the principle gives no information about the rotation of the material points and the principal directions of the strain ellipsoid, the deformed state of the body is indeterminable, in general, using (2.1).
Note that in the linearized theory of elasticity, the infinitesimal rotations can be uniquely determined from the strain tensor and, after taking into account the inverse stress-strain relations, (2.1) is equivalent to Castigliano’s variational principle.

2.2. The generalized principle of complementary virtual work in the current configuration. The basic idea in the generalization of the principle of complementary virtual work is to introduce the finite rotations as independent variables into (2.1). This can be accomplished if the Cauchy stress tensor is not considered to be a priori symmetric. Enforcing the rotational equilibrium equations for the Cauchy stresses into the principle is, however, not as straightforward as in the linear case, where the (infinitesimal) rotation tensor is skew-symmetric; it needs the utilization of the fact that for any orthogonal tensor \( \mathbf{R} \), the tensor \( \delta \mathbf{R} \cdot \mathbf{R}^T \) is skew-symmetric. Then the generalized principle of complementary virtual work can be stated as follows. Let equality

\[
\int_\Omega \left[ (1 - \mathbf{R}^T \cdot \mathbf{V}^{-1}) : \delta \mathbf{\tau} + \mathbf{\tau} : (\delta \mathbf{R} \cdot \mathbf{R}^T) \right] \, d^4 \Omega = \int_{\Gamma_u} \mathbf{u} \cdot \delta \mathbf{\tau} \cdot \mathbf{'} \mathbf{n} \, d^1 \Gamma \tag{2.5}
\]

hold for all statically admissible (not a priori symmetric) Cauchy stresses \( \delta \mathbf{\tau} \) satisfying (2.2) and (2.3), and for all \( \delta \mathbf{R} \) obtained from an orthogonal rotation tensor \( \mathbf{R} \), where \( \mathbf{V} \) is an arbitrary but symmetric tensor. Then the tensor

\[
\mathbf{D} := 1 - \mathbf{R}^T \cdot \mathbf{V}^{-1}
\]

is the displacement gradient tensor, \( \mathbf{R}^T \cdot \mathbf{V}^{-1} \) is the inverse deformation gradient, and the Cauchy stress tensor \( \mathbf{\tau} \) is symmetric.

To prove the above statements it should be taken into account that orthogonality of \( \mathbf{R} \) implies

\[
\delta (\mathbf{R} \cdot \mathbf{R}^T) = 0, \quad \delta \mathbf{R} \cdot \mathbf{R}^T = -\mathbf{R} \cdot \delta \mathbf{R}^T = - (\delta \mathbf{R} \cdot \mathbf{R}^T)^T, \tag{2.7}
\]

which means that, as indicated above, the tensor

\[
\delta \mathbf{\Theta} := \delta \mathbf{R} \cdot \mathbf{R}^T
\]

appearing in the volume integral of (2.5) is skew-symmetric. Using the above notations, (2.5) can be rewritten in the following brief form:

\[
\int_\Omega (\mathbf{D} : \delta \mathbf{\tau} + \mathbf{\tau} : \delta \mathbf{\Theta}) \, d^4 \Omega = \int_{\Gamma_u} \mathbf{u} \cdot \delta \mathbf{\tau} \cdot \mathbf{'} \mathbf{n} \, d^1 \Gamma. \tag{2.9}
\]

In the course of integral transformations it should be taken into account that

- translational equilibrium (2.2) for the (non-symmetric) Cauchy stresses \( \delta \mathbf{\tau} \) can be satisfied a priori by introducing a tensor of first-order stress functions \( \mathbf{\chi} \) as

\[
\delta \mathbf{\tau} = (\text{curl} \delta \mathbf{\chi})^T \quad \mathbf{x} \in \mathring{\Omega}, \tag{2.10}
\]

where only six out of the nine components of \( \delta \mathbf{\chi} \) are independent and the other three components can be set to zero;
• the homogeneous stress boundary condition (2.3) for \( \delta \tau \) in terms of first-order stress functions takes the form

\[
\text{div}_x \delta \chi = 0 \quad x \in \Gamma_\sigma,
\]

(2.11)

where \( \delta \chi \) on \( \Gamma_\sigma \) can be obtained as the gradient of an arbitrary vector \( \mathbf{v} \) defined on \( \Gamma_\sigma \) as

\[
\delta \chi = \text{grad} \mathbf{v} \quad x \in \Gamma_\sigma,
\]

(2.12)

provided equation

\[
(\delta \chi - \text{grad} \mathbf{v}) \cdot 'l = 0 \quad x \in \ell
\]

(2.13)

holds on the common curve \( \ell \) of boundary surfaces \( \Gamma_\sigma \) and \( \Gamma_u \) with \( 'l \) being the tangent unit vector to \( \ell \) [17];

• the arbitrary but skew-symmetric tensor \( \delta \Theta \) defined by (2.8) can be written in terms of its vector \( \delta \theta \) as

\[
\delta \Theta = 1 \times \delta \theta,
\]

(2.14)

where the three components of \( \delta \theta \) are also arbitrary.

Applying the Gauss- and Stokes-theorem and taking into account (2.10)-(2.14), (2.9) can be transformed into

\[
\int_{\Omega} \left[ -(\text{curl} \mathbf{D})^T : \delta \chi + \tau : (1 \times \delta \theta) \right] \ d'\Omega - \int_{\Gamma_\sigma} \left( \text{div}_x \mathbf{D} \right) \cdot \mathbf{v} \ d'\Gamma
\]

\[
+ \int_{\Gamma_u} \left[ (\mathbf{D} - \text{grad} \tilde{\mathbf{u}}) \times 'n \right] : \delta \chi \ d'\Gamma - \oint_{\ell} \mathbf{v} \cdot (\mathbf{D} - \text{grad} \tilde{\mathbf{u}}) \cdot 'l \ d's = 0.
\]

(2.15)

Equation (2.15) holds for all \( \delta \chi, \delta \theta \) and \( \mathbf{v} \), which means that their coefficients should be equal to zero. This condition implies the following independent Euler-Lagrange equations and natural boundary conditions of the generalized principle of complementary virtual work [18]:

the first-order compatibility equations (six independent equations)

\[
(\text{curl} \mathbf{D})^T = 0 \quad x \in \Omega,
\]

(2.16)

the symmetry of the Cauchy stress tensor (three equations)

\[
\tau - \tau^T = 0 \quad x \in \Omega,
\]

(2.17)

the first-order compatibility boundary conditions (three equations)

\[
\text{div}_x \mathbf{D} = 0 \quad x \in \Gamma_\sigma,
\]

(2.18)

the strain boundary conditions (six independent equations)

\[
(\mathbf{D} - \text{grad} \tilde{\mathbf{u}}) \times 'n = 0 \quad x \in \Gamma_u,
\]

(2.19)
and the continuity condition
\[(D - \text{grad } \hat{u}) \cdot 'l = 0 \quad x \in '\ell\]

on the common curve 'l of 'Γ_u and 'Γ_\sigma, where 'l is the unit tangent to the curve 'l (Figure 1). It can be pointed out that condition (2.19) on 'Γ_u implies compatibility boundary condition div_x D = 0 on 'Γ_u as well. Since compatibility equations (2.16) are satisfied in 'Ω and the compatibility boundary conditions are satisfied on the whole boundary surface 'Γ, tensor \( D = 1 - R^T \cdot V^{-1} \) is the displacement gradient tensor, \( R^T \cdot V^{-1} \) is the inverse deformation gradient tensor and, following from the polar decomposition theorem, R and V are the rotation tensor and the left stretch tensor, respectively.

The independent equation system of non-linear elasticity in terms of the Cauchy stress tensor \( \tau \) and the orthogonal rotation tensor R consists of the translational equilibrium equations \( \text{div } \tau + \text{rot } b = 0 \), rotational equilibrium equations (2.17) and compatibility equations (2.16) as field equations, as well as stress boundary conditions \( \tau \cdot 'n = 'p \), compatibility boundary conditions (2.18), and the boundary conditions for the strain tensor, (2.19) and (2.20). This equation system or, equivalently, the generalized principle of complementary virtual work with functional (2.5) can be used for solving non-linear elasticity problems only in that case, however, when invertible constitutive equation between the Cauchy stress tensor \( \tau \) and the left stretch tensor \( V \) exists. Such a constitutive relation for isotropic materials can be given in the form
\[\tau(V) = a_0 I + a_1 V + a_{-1} V^{-1},\]

where \( a_0, a_1, a_{-1} \) are functions of the scalar invariants of V [19][15].

**Remark 1.** When compatibility equations (2.16), (2.18)-(2.19) hold, D is the gradient of an arbitrary vector field denoted by u, i.e. \( D = \text{grad } u \). Assuming that D is given or obtained using the generalized complementary virtual work theorem, vector field \( u(P) \) at point P of the elastic body can be computed from D as
\[u(P) = u(O) + \int_O^P D \cdot dx = u(O) + \int_O^P \text{grad } u \cdot dx,\]

provided the value \( u(O) \) at an arbitrary point O of the body is known (the integral is path-independent). If \( u(O) \) is the displacement of point \( O \in '\Gamma_u \), then \( u(x) \) is the displacement field of the elastic body.

**Remark 2.** The inner product \( \tau : (\delta R \cdot R^T) \) in (2.5) can be considered as a Lagrange-multiplier term enforcing the symmetry condition for the Cauchy stress tensor into the principle. If \( \tau \) is a priori symmetric, this term disappears from (2.5) and the classical principle of complementary virtual work (2.1) is obtained.

**Remark 3.** In the dual formulation of the linearized theory of micropolar elasticity, where the use of both second- and first-order stress functions are needed, the correct number of first-order stress functions have been used by Kozák-Szeidl [20].
3. The generalized principle of complementary virtual work in the reference configuration

3.1. Transformations between the current and reference configurations.

The generalized principle of complementary virtual work in the reference configuration can be derived from its form (2.5) valid in the current configuration. As a first step, relation between the variations of the Cauchy stress tensor $\delta \tau$ and the first Piola-Kirchhoff stress tensor $\delta T$ should be taken into account [6]:

$$\delta \tau = J^{-1} \delta T : F^T,$$

where, assuming that (2.5) holds, $F$ is the deformation gradient tensor and $J$ is the Jacobian of the deformation gradient. Making use of (3.1) as well as the Piola transformation [21], requirements of statically admissibility for $\delta \tau$, (2.2)-(2.3), are equivalent to equilibrium equations

$$\text{Div} \delta T = 0 \quad X \in \Omega$$

and stress boundary conditions

$$\delta T \cdot \mathbf{n} = 0 \quad X \in \Gamma_{\sigma}$$

for the first Piola-Kirchhoff stress tensor $T$. The next step is to transform the integrands of functional (2.5) into the reference configuration by taking into account the relations between the surface and volume elements of the current and reference configurations [6]:

$$d'\Omega = J d\Omega,$$

and

$$'\mathbf{n} d'T = J F^{-T} : \mathbf{n} d\Gamma.$$

Then, using (3.1) and (3.4)-(3.5), integrands of (2.5) can be transformed as follows:

$$1 : \delta \tau d'\Omega = \delta T : F \ d\Omega,$$

$$(\mathbf{R}^T \cdot V^{-1}) : \delta \tau d'\Omega = F^{-1} : (\delta T \cdot F) d\Omega = \delta T : 1 d\Omega,$$

$$\tau : (\delta \mathbf{R} \cdot \mathbf{R}^T) d'\Omega = (\mathbf{T} \cdot \mathbf{F}^T) : (\delta \mathbf{R} \cdot \mathbf{R}^T) d\Omega,$$

$$\mathbf{u}(x) \cdot \delta \tau : '\mathbf{n} d'T = \mathbf{u}(X) \cdot \delta T : \mathbf{n} d\Gamma.$$

Inserting (3.6)-(3.9) in (2.5), the first form of the generalized principle of complementary virtual work in the reference configuration reads:

$$\int_{\Gamma_{\Omega}} \left[ \delta T : (\mathbf{F} - 1) + (\mathbf{T} \cdot \mathbf{F}^T) : (\delta \mathbf{R} \cdot \mathbf{R}^T) \right] d\Omega = \int_{\Gamma_{\Gamma_{\sigma}}} \mathbf{u} \cdot \delta T : \mathbf{n} d\Gamma.$$ (3.10)

If equality (3.10) holds for all statically admissible first Piola-Kirchhoff stresses $\delta T$ and all rotations $\delta \mathbf{R}$ (obtained from an arbitrary but orthogonal $\mathbf{R}$), then the tensor $\mathbf{H} := \mathbf{F} - 1$ is the displacement gradient tensor in the reference configuration, $\mathbf{F}$ is the deformation gradient and the product tensor $\mathbf{T} \cdot \mathbf{F}^T$ is symmetric. Following from
the relation between the Cauchy stresses and the first Piola-Kirchhoff stresses \([22]\), the latter condition is equivalent to the symmetry of the Cauchy stress tensor \(\tau\).

Proof of the above statements can be accomplished in similar steps to those presented in Section 2.2. However, the principle of complementary virtual work in its form \((3.10)\) is still not applicable to the (numerical) solution of general non-linear elasticity problems, as the constitutive relation between the first Piola-Kirchhoff stress tensor and the displacement gradient (or deformation gradient) tensor is not uniquely invertible \([9][11]\).

To obtain the most useful form of the principle of complementary virtual work in the reference configuration, polar decompositions of tensors \(F\) and \(T\) should be taken into account:

\[
F = R \cdot U, \quad T = R \cdot \sigma, \tag{3.11}
\]

where \(\sigma\) is the Biot stress tensor \([23][11]\). Making use of \((3.11)\), integrands on the left-hand-side of \((3.10)\) can be transformed as follows:

\[
\delta T : F = \delta T : (R \cdot U) = \delta \sigma : U + (\delta R^T \cdot R) : (\sigma \cdot U), \tag{3.12}
\]

\[
(T \cdot F^T) : (\delta R \cdot R^T) = (R^T \cdot \delta R) : (\sigma \cdot U). \tag{3.13}
\]

On inserting \((3.12)-(3.13)\) in \((3.10)\) and considering that in the reference configuration orthogonality of \(R\) implies

\[
\delta (R^T \cdot R) = 0, \quad \delta R^T \cdot R = -R^T \cdot \delta R = -(\delta R^T \cdot R)^T, \tag{3.14}
\]

by which

\[
(R^T \cdot \delta R) : (\sigma \cdot U) + (\delta R^T \cdot R) : (\sigma \cdot U) = 0, \tag{3.15}
\]

the final form of the generalized principle of complementary virtual work in the reference configuration is obtained:

\[
\int_{\Omega} (\delta \sigma : U - \delta T : 1) \, d^o \Omega = \int_{\partial \Omega} \tilde{u} \cdot \delta T \cdot \hat{n} \, d^o \Gamma. \tag{3.16}
\]

If equality \((3.16)\) holds (independently of the above derivation) for all statically admissible first Piola-Kirchhoff stresses \(\delta T\) and all rotations \(\delta R\) together with relation \(\delta \sigma = \delta (R^T \cdot T)\), where \(U\) is a symmetric tensor, then \(H := R \cdot U - I\) is the displacement gradient tensor, \(F := R \cdot U\) is the deformation gradient, and the co-rotated Kirchhoff stress tensor \(K := \sigma \cdot U\) is symmetric, which implies the symmetry of the Kirchhoff- and Cauchy stress tensor as well. The proof of the above statements is given in Section 3.2 Note that due to the symmetry of \(U\), \((3.16)\) depends on the symmetric part of the Biot stress tensor, \(\sigma_s = (\sigma + \sigma^T)/2 = (R^T \cdot T + T^T \cdot R)/2\), which is sometimes referred to as Jaumann stress tensor \([8][9]\).

**Remark 4.** The generalized principle of complementary virtual work is independent of the actual constitutive equations. Since \(U\) and \(\sigma_s\) are objective stress and strain measures, single-valued, invertible constitutive relation exists between them. Assuming hyperelastic materials, it can be given in the form

\[
U(\sigma_s) = \frac{\partial W_e(\sigma_s)}{\partial \sigma_s}, \tag{3.17}
\]
where $W_c(\sigma_s)$ is the complementary strain energy density. Using (3.17), inner product $\delta\sigma : U$ on the left-hand-side of (3.16) can be written as

$$\delta\sigma : U = \delta\sigma_s : U = \delta\sigma_s : \frac{\partial W_c(\sigma_s)}{\partial \sigma_s} = \delta W_c(\sigma_s).$$

On inserting (3.18) in (3.16), we obtain the zero-valued first variation of Fraeij's de Veubeke's dual-mixed variational principle in terms of the first Piola-Kirchhoff stress tensor and the orthogonal rotation tensor [7]:

$$\delta F(\delta T, \delta R) = \int_{\Omega} \left[ \delta W_c(\sigma_s) - \delta T : 1 \right] d\Omega - \int_{\Gamma_n} \mathbf{u} \cdot \delta T \cdot \mathbf{n} : d\Omega = 0.$$  

This result indicates that the generalized principle of complementary virtual work in the reference configuration is equivalent to the two-field dual-mixed principle of Fraeij's de Veubeke, provided constitutive equations of type (3.17) are being taken into account. □

3.2. Independent equation system of non-linear elasticity in the reference configuration. The independent equation system of non-linear elasticity in terms of the first Piola-Kirchhoff stress tensor $T$ and the orthogonal rotation tensor $R$ can be derived from (3.16) by taking into account subsidiary conditions of statically admissibility, (3.2) and (3.3), for $\delta T$, the orthogonality condition for $R$, as well as the symmetry condition for $U$. In the course of integral transformations it should be taken into account that [18]

- following from (3.11), variation of the Biot stress tensor can be expressed by variations of $T$ and $R$ as

$$\delta\sigma = \delta R^T \cdot T + \delta T^T \cdot R,$$

- equilibrated $\delta T$ satisfying (3.2) can be obtained from an arbitrary tensor of first-order stress functions $\Psi$ as

$$\delta T = (\text{Curl } \delta \Psi)^T,$$  

where only six out of the nine components of $\delta \Psi$ are independent and the other three components can be set to zero;

- orthogonality of $R$ and (3.14) implies that tensor

$$\delta\phi := R^T \cdot \delta R,$$

is skew-symmetric, from which first variation of the rotation tensor can be expressed as

$$\delta R = R \cdot \delta\phi;$$

- the homogeneous stress boundary condition (3.3) for $\delta T$ in terms of first-order stress functions takes the form

$$\text{Div}_r \delta \Psi = 0 \quad X \in \mathbb{T}_\sigma,$$
where $\delta \Psi$ on $^\theta \Gamma_\sigma$ can be obtained as the gradient of an arbitrary vector $w$ defined on $^\theta \Gamma_\sigma$ as

$$
\delta \Psi = \text{Grad} \ w \quad X \in \ ^\theta \Gamma_\sigma,
$$

(3.25)

provided equation

$$
(\delta \Psi - \text{Grad} \ w) \cdot \ ^\theta l = 0 \quad X \in \ ^\theta \ell
$$

(3.26)

holds on the common curve $^\theta \ell$ of boundary surfaces $^\theta \Gamma_\sigma$ and $^\theta \Gamma_u$ with $^\theta l$ being the tangent unit vector to $^\theta \ell$;

- the arbitrary but skew-symmetric tensor $\delta \varphi$ defined by (3.22) can be written in terms of its vector $\delta \varphi$ as

$$
\delta \varphi = 1 \times \delta \varphi,
$$

(3.27)

where the three components of $\delta \varphi$ are also arbitrary.

Introducing the notation

$$
H := R \cdot U - 1
$$

(3.28)

and applying the Gauss- and Stokes-theorem and taking into account (3.20)–(3.27), (3.16) can be transformed into

$$
\int_{^\theta \Omega} \left[ -(\text{Curl} \ H)^T : \delta \Psi - (\sigma \cdot U) : (1 \times \delta \varphi) \right] d^\theta \Omega - \int_{^\theta \Gamma_\sigma} (\text{Div}_r \ H) \cdot w \ d^\theta \Gamma

+ \int_{^\theta \Gamma_u} [(H - \text{Grad} \tilde{u}) \times ^\theta n] : \delta \Psi d^\theta \Gamma - \int_{^\theta \ell} w \cdot (H - \text{Grad} \tilde{u}) \cdot ^\theta l d^\theta s = 0.
$$

(3.29)

Equality (3.29) holds for all $\delta \Psi$, $\delta \varphi$ and $w$, which means that their coefficients should be equal to zero. The independent Euler-Lagrange equations and natural boundary conditions of the generalized principle of complementary virtual work in the reference configuration are [18]:

- the first-order compatibility equations for $H$ (six independent equations):

$$
(\text{Curl} \ H)^T = 0 \quad X \in \ ^\theta \Omega,
$$

(3.30)

- the symmetry condition for the co-rotated Kirchhoff stress tensor $K = \sigma \cdot U$ (three equations):

$$
\sigma \cdot U = U \cdot \sigma^T \quad X \in \ ^\theta \Omega,
$$

(3.31)

- the first-order compatibility boundary conditions (three equations):

$$
\text{Div}_r \ H = 0 \quad X \in \ ^\theta \Gamma_\sigma,
$$

(3.32)

- the strain boundary conditions (six independent equations):

$$
(H - \text{Grad} \tilde{u}) \times ^\theta n = 0 \quad X \in \ ^\theta \Gamma_u,
$$

(3.33)
and the continuity condition
\[ (H - \text{Grad} \tilde{u}) \cdot \cdot l = 0 \quad X \in \cdot l \] (3.34)
on the common curve \( \cdot l \) of surface parts \( \cdot \Gamma_u \) on \( \cdot \Gamma_{\sigma} \), where \( \cdot l \) is the unit tangent to the curve \( \cdot l \) (Figure 1). Condition (3.33) implies compatibility boundary condition \( \text{Div}_v H = 0 \) on \( \cdot \Gamma_u \), i.e., compatibility boundary conditions are satisfied on the whole surface \( \cdot \Gamma \). This fact, together with compatibility equations (3.30) means that tensor field \( H \), defined by (3.28), is the displacement gradient in the reference configuration and \( R \cdot U = H + l \) is the deformation gradient. With the above derivation, statements in Section 3.1 regarding the complementary virtual work theorem in the reference configuration have been proven.

The independent equation system of non-linear elasticity in terms of the first Piola-Kirchhoff stress tensor \( T \) and the orthogonal rotation tensor \( R \) consists of the translational equilibrium equations \( \text{Div} \cdot T + \cdot \rho = 0 \), compatibility equations (3.30) and, taking into account relation \( \sigma = R^T \cdot T \), symmetry condition (3.31) as field equations, as well as stress boundary conditions \( T \cdot \cdot n = \cdot \tilde{p} \), compatibility boundary conditions (3.32), and boundary conditions for the displacement gradient tensor, (3.33) and (3.34). This equation system or, equivalently, the generalized principle of complementary virtual work with functional (3.16) should, of course, be used together with constitutive equations of type (3.17) when solving non-linear elasticity problems.

In connection with the above derived equation system it should be noted that a derivation has already been given by Fraeijs de Veubeke [7]. However, his derivation is incomplete in the following sense: (i) The nine compatibility equation of [7] is not independent, because all the nine components of the first-order stress function tensor were used (instead of six) in the derivation. (ii) Stress boundary condition (3.24) have been taken into account by the use of a Lagrange-multiplier which was a priori assumed (incorrectly) to be the displacement field on the boundary part \( \cdot \Gamma_{\sigma} \). Due to this assumption, compatibility boundary conditions (3.32) and strain boundary conditions (3.33) did not appear in the equation system of [7] (instead of (3.32), an identity was obtained), and, furthermore, no fitting condition for the Lagrange multiplier and the prescribed displacement field \( \tilde{u} \) on the curve \( \cdot l \) was obtained (a condition which would be equivalent to (3.34)).

**Remark 5.** In geometrically non-linear elasticity problems, instead of \( U \) the use of the Jaumann strain tensor \( \varepsilon = U - I \) is more practical (due to the linear Constitutive relations between the Biot stresses and Jaumann strains in that case). Then, taking into account that from (3.20) we have
\[ \cdot \sigma : 1 = \cdot \delta R : T + \cdot \delta T : R, \] (3.35)
the generalized principle of complementary virtual work can be transformed into the following form:
\[ \int_{\cdot \Omega} \{ \cdot \sigma : \varepsilon + \cdot \delta T : (R - 1) + \cdot \delta R : T \} d \cdot \Omega = \int_{\cdot \Gamma_u} \tilde{u} \cdot \cdot \delta T : \cdot n : d \cdot \Gamma. \] (3.36)
Incremental form of this principle and its consistent linearization can be found in [18].

\( \square \)
4. Concluding remarks

Despite the fact that stress-based methods and numerical models are usually more involved, both theoretically and numerically, than displacement-based methods and models, stress-based formulations receive an emerging interest in computational mechanics. The practical importance of such models and methods in the numerical analysis and solution of linear and non-linear elasticity problems can probably be traced back to the numerical difficulties and convergence problems encountered in the application of the classical displacement-based finite element methods to parameter dependent or constraint problems of elasticity (and plasticity).

Theoretical developments presented in this paper has been motivated by the limited applicability of the stress-based principle of complementary virtual work in non-linear elasticity. A modification of the classical principle is proposed by introducing the orthogonal rotation tensor into the principle. In the current configuration this is done by incorporating the rotational equilibrium for the Cauchy stress tensor into the principle. Applicability of the generalized principle requires, however, the existence of an invertible constitutive relation between the Cauchy stress tensor and the left stretch tensor. From the numerical point of view, it is more useful to transform the principle into the reference configuration, where, provided that constitutive equations between the Biot stress and right stretch tensors are being taken into consideration, the principle is equivalent to the two-field, complementary energy-based variational principle of Fraeijs de Veubeke.

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