STABILITY OF A CIRCULAR PLATE STIFFENED WITH A CYLINDRICAL SHELL

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1. INTRODUCTION

In the engineering practice stability problems of plates loaded in their own plane are especially interesting ones. To the author's knowledge paper [1, 1981] by Brian was the first one which dealt with the stability problem of a circular plate. Since then a number of papers have been devoted to this issue. Here we have cited only a few [2, 1933], [3, 1984] and remark that further references can be found in the papers cited.

A circular plate can be stiffened in various ways. For example we can apply a corrugation to it, or it can be stiffened by a cylindrical shell attached to the plate on its boundary. The present paper investigates the stability of a circular plate provided that the plate is stiffened by a cylindrical shell. This problem was partly solved by Szilassy [4, 1971], [5, 1976] who set up a differential equation for the rotation field and solved the corresponding eigenvalue problem under the assumption that the shell is subjected to a constant radial load in the middle plane of the plate.

First we shall consider the governing equations of the stability problem if there is no stiffening. Then we shall clarify what conditions are to be satisfied on the circle where the middle surfaces of the plate and shell meet. By solving the differential equations set up both for the shell and for the plate in terms of the displacements we derive a non-linear equation from the eigenvalue problem to be solved. We shall also present the results of our computations.

2. THE STRUCTURE AND THE LOADS APPLIED

Figure 1 shows the geometry of a structure consisting of a circular plate and cylindric shell. We shall assume that there is no hole in the plate, i.e., $R_i = 0$. We shall also assume that the plate and the shell are thin, consequently we can apply the Kirchhoff theory of plates and shells. If the shell and plate are made of the same isotropic material, then E and ν are the Young-module and the Poisson ratio, respectively. We consider two loads: 1. a constant radial load in the middle plane of the plate (see Fig. 1); 2. a constant uniform load exerted on the external lateral surface of the shell. The loads are rigid, i.e. they keep their original direction.

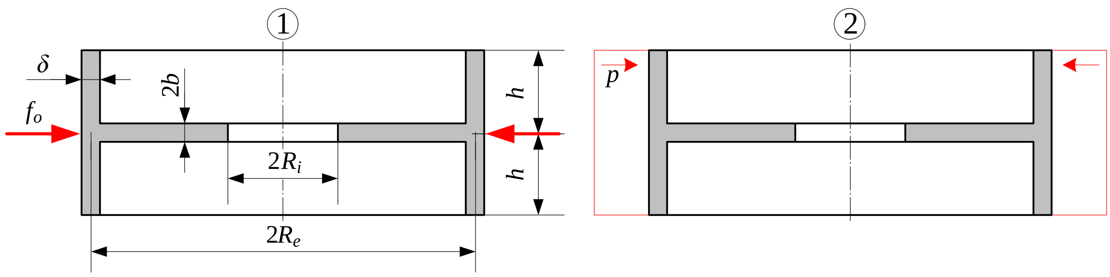


FIGURE 1.

Supposing small and elastic axisymmetric deformations, we determine the critical load of the structure. The results obtained shall clarify what effect the stiffening has on the critical load.

3. CONDITIONS BETWEEN THE TWO STRUCTURAL ELEMENTS

When we examine the conditions to be satisfied on the intersection line of the middle surfaces of the shell and the plate we shall assume that this line is a circle with radius $R = R_e$ where R_e is the external radius of the plate and at the same time we regard this radius as if it were that of the middle surface of the shell. We shall also assume that due to the load the shell and plate deform together on this line that is the displacement and the rotation are the same both for the plate and for the shell at $R = R_e$.

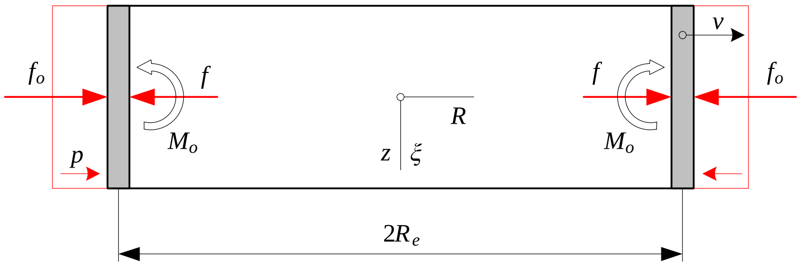


FIGURE 2.

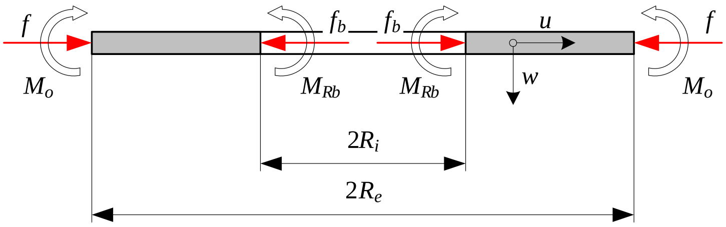


FIGURE 3.

Separated from each other mentally the two structural elements (the plate and the shell) are shown in Figures 2 and 3, where we can also see the load for loading case 1 and the inner force system. We can calculate the intensity of the distributed forces f and that of the distributed couple system M_o from the conditions mentioned above, that is from the fact that (a) the radial displacement u of the plate is equal to the radial displacement v of the shell, the latter is perpendicular to its middle surface; (b) the rotations are also the same. Consequently the following conditions are to be satisfied

$$u\big|_{R=R_e} = u_o = v_o = v\big|_{\xi=0} \tag{1a}$$

and

$$\vartheta_o = -\frac{\mathrm{d}w}{\mathrm{d}R}\bigg|_{R=R_e} = \frac{\mathrm{d}v}{\mathrm{d}\xi}\bigg|_{\xi=0} . \tag{1b}$$

It is also obvious that the shear force Q_R in the plate is zero on the intersection line of the two middle surfaces:

$$Q_R\big|_{R=R_e} = 0 . \tag{1c}$$

4. DEFORMATION OF THE CYLINDRIC SHELL

If the shell is subjected to a constant radial load p directed towards the axis of the shell [6, 1967] and the deformation of the middle surface is axisymmetric, the radial displacement v on the middle surface – see Figure 2 – should satisfy the differential equation

$$\frac{\mathrm{d}^4 v}{\mathrm{d}\xi^4} + 4\beta^4 v = \frac{1}{I_{1h}E_{1h}}\left(-p - \nu\frac{N_\xi}{R_K}\right) \tag{2a}$$

where

$$\beta = \nu_o \sqrt{\frac{R_e}{\delta}\frac{1}{R_e}}\,, \quad \nu_o = \sqrt[4]{3(1-\nu^2)}\,, \quad I_{1h} = \delta^3/12\,, \quad E_1 = E_{1h} = E/(1-\nu^2) \tag{2b}$$

and under the present conditions $N_\xi = 0$. The solution of equation (2a) assumes the form

$$v(\xi) = \sum_{i=1}^{4} a_i V_i(\beta\xi) + v_p\,; \quad v_p = -p/\beta^4 I_1 E_1 \tag{3}$$

where V_i $(i = 1, \ldots, 4)$ are Krylov-functions and a_i $(i = 1, \ldots, 4)$ are integration constants. For a cylindric shell Q_ξ is the shear force and M_ξ is the bending moment.

The solution for v is the superposition of the solutions we shall determine for the following two partial loads:

Load 1.: The shell is subjected to the distributed forces f_o and f. The corresponding boundary conditions are as follows:

$$Q_\xi|_{\xi=0} = -\frac{f_o - f}{2}\,, \quad \left.\frac{\mathrm{d}v}{\mathrm{d}\xi}\right|_{\xi=0} = 0\,, \quad Q_\xi|_{\xi=h} = 0\,, \quad M_\xi|_{\xi=h} = 0\,. \tag{4a}$$

Load 2.: The shell is subjected to the couple system $M_\circ$. Now we have the following boundary conditions:

$$v(\xi)|_{\xi=0} = 0\,, \quad M_\xi|_{\xi=0} = -\frac{M_o}{2}\,, \quad Q_\xi|_{\xi=h} = 0\,, \quad M_\xi|_{\xi=h} = 0\,. \tag{4b}$$

Observe that the boundary conditions at $\xi = h$ are the same for the two partial loads.

When calculating the partial solutions we utilize that the sear forces Q_ξ and the bending moment M_ξ can be determined from the relations

$$Q_\xi = I_{1h}E_{1h}\frac{\mathrm{d}^3 v}{\mathrm{d}\xi^3} \qquad \text{and} \qquad M_\xi = -I_{1h}E_{1h}\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2}\,. \tag{5}$$

After some hand made calculations in which we have made use of the definitions of the Krylov-functions we obtain

$$v_o = v|_{\xi=0} = -\underbrace{\frac{\nu_o}{2E}\left(\frac{R_k}{\delta}\right)^{\frac{3}{2}} \frac{\cos 2h\beta + \cosh 2h\beta + 2}{\sin 2h\beta + \sinh 2h\beta}}_{\alpha}(f_o - f) = -\alpha\,(f_o - f) \tag{6}$$

and

$$\vartheta_o = \left.\frac{dv}{d\xi}\right|_{\xi=0} = -\underbrace{\frac{\nu_o^3}{E}\left(\frac{R_k}{\delta}\right)^{\frac{1}{2}} \frac{\cos 2h\beta + \cosh 2h\beta + 2}{\sinh 2h\beta - \sin 2h\beta}\frac{1}{\delta^2}}_{\kappa}M_o = -\kappa\, M_o\,. \tag{7}$$

where α and κ are defined by the relations above. The radial displacement for the constant radial load p is naturally a bit different:

$$v(\xi = 0) = \frac{\nu_o}{2E}\left(\frac{R_K}{\delta}\right)^{\frac{3}{2}} \frac{\cos 2\beta h + \cosh 2\beta h + 2}{\sin 2\beta h + \sinh 2\beta h} f - \underbrace{\frac{1}{E}\frac{R_K^2}{\delta}}_{\varphi} p = \alpha f - \varphi p \tag{8}$$

5. DEFORMATION OF THE CIRCULAR PLATE

In an axissymmetric case every physical quantity depend only on the radius R. Consequently the differential equation for the displacement w takes the form:

$$\Delta_H \Delta_H w - \frac{1}{I_1 E_1}\left[N_R \frac{\mathrm{d}^2 w}{\mathrm{d}R^2} + N_\varphi \frac{1}{R}\frac{\mathrm{d}w}{\mathrm{d}R}\right] = \frac{1}{I_1 E_1} p_z \tag{9a}$$

where $p_z = 0$ (there is no load on the plate in the direction z) and

$$\Delta_H = \frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R} = \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}}{\mathrm{d}R}\right) . \tag{9b}$$

The inner forces N_R and N_φ in the circular plate due to the in-plane load are

$$N_R = -A + \frac{B}{R^2} \qquad \text{and} \qquad N_\varphi = -A - \frac{B}{R^2} . \tag{10}$$

where A and B are integration constants. If there is no hole in the plate then

$$A = f \qquad \text{and} \qquad B = 0 . \tag{11}$$

The radial displacement due to the load f exerted on the outer diameter can be calculated from the following equation

$$v_o = -K\frac{R_k}{2b}\frac{f}{E} \quad \text{where} \quad K = 1 - \nu . \tag{12}$$

If we introduce the dimensionless independent variable $\rho = R/R_e$ from equation (9a) we obtain

$$\tilde{\Delta}\tilde{\Delta} w + \mathfrak{F}\tilde{\Delta} w = 0 \tag{13a}$$

in which

$$\mathfrak{F} = R_K^2 \frac{f}{I_1 E_1} \qquad \text{and} \qquad \Delta_H = \frac{1}{R_k^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\right) = \frac{1}{R_k^2}\tilde{\Delta} \tag{13b}$$

Solution for the above equation is of the form

$$w(\rho) = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4 , \tag{14a}$$

$$Z_1 = 1 , \qquad Z_2 = \ln\rho , \qquad Z_3 = J_o(\sqrt{\mathfrak{F}}\rho) , \qquad Z_4 = Y_o(\sqrt{\mathfrak{F}}\rho) , \tag{14b}$$

where c_i, $(i = 1, \ldots, 4)$ are integration constants while Z_i are the independent particular solutions. For small values of $\sqrt{\mathfrak{F}}\rho$ it holds the following asymptotic relation

$$Y_o(\sqrt{\mathfrak{F}}\rho) = 2\ln(\sqrt{\mathfrak{F}}\rho)/\pi . \tag{15}$$

The displacement w should be limited if $\rho \to 0$. If we take into account equation (15) and the relation $Z_3(0) = J_o(0) = 1$ we obtain that w is limited if

$$c_2 = -2c_4/\pi . \tag{16}$$

Consequently

$$w(\rho) = c_1 + c_4\left[Y_o(\sqrt{\mathfrak{F}}\rho) - \frac{2}{\pi}\ln(\sqrt{\mathfrak{F}}\rho)\right] + c_3 J_o(\sqrt{\mathfrak{F}}\rho) . \tag{17}$$

In what follows use will be made of the relations

$$2J_1(x)/x = J_2(x) + J_o(x) \qquad 2Y_1(x)/x = Y_2(x) + Y_o(x) \tag{18}$$

and

$$(\tilde{\Delta} + \mathfrak{F})(c_3 J_o(\sqrt{\mathfrak{F}}\rho) + c_4 Y_o(\sqrt{\mathfrak{F}}\rho)) = 0 , \tag{19}$$

together with the equation

$$Q_R = I_1 E_1 \frac{\mathrm{d}}{\mathrm{d}R}\Delta_H w + N_R \frac{\mathrm{d}w}{\mathrm{d}R}\,, \tag{20}$$

which provides us the shear force – here $I_1 = 8b^3/12$. It is not too difficult to check that

$$Q_R \frac{R_K^3}{I_1 E_1} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \mathfrak{F}\right]\left[c_1 - c_4\frac{2}{\pi}\ln(\sqrt{\mathfrak{F}}\rho)\right] = -c_4\frac{2}{\pi}\mathfrak{F}\frac{1}{\rho}\,. \tag{21}$$

Since the shear force Q_R is zero on the circle with radius $R = \rho R_K$ if $R \to 0$ (or which is the same if $\rho \to 0$) we obtain from the above equation that

$$2\pi R\, Q_R = -4c_4 f = 0 \quad \mapsto \quad c_4 = 0\,. \tag{22}$$

Therefore the solution for w is of the form

$$w(\rho) = c_1 + c_3 J_o(\sqrt{\mathfrak{F}}\rho)\,. \tag{23}$$

The integration constants left can be calculated from the conditions to be satisfied on the intersection line of the two middle surfaces for which $\rho_e = 1$. After some manipulations the left side of condition (7) takes the form

$$\vartheta_o = -\left.\frac{\mathrm{d}w}{\mathrm{d}R}\right|_{R=R_e} = -\left.\frac{1}{R_e}\frac{\mathrm{d}w}{\mathrm{d}\rho}\right|_{\rho=1} = -\left. c_3\frac{1}{R_e}\frac{\mathrm{d}J_o(\sqrt{\mathfrak{F}}\rho)}{\mathrm{d}\rho}\right|_{\rho=1} = c_3\frac{1}{R_e}\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}})\,. \tag{24a}$$

At the same time the right side can be manipulated into the form

$$\begin{aligned} M_o = -I_1E_1\left.\left[\frac{\mathrm{d}^2 w}{\mathrm{d}R^2} + \nu\frac{1}{R}\frac{\mathrm{d}w}{\mathrm{d}R}\right]\right|_{R=R_e} &= -c_3\frac{I_1E_1}{R_e^2}\left\{\frac{\mathfrak{F}}{2}\left[J_2(\sqrt{\mathfrak{F}}\rho) - J_o(\sqrt{\mathfrak{F}}\rho)\right] - \right.\\ \left. - \frac{\nu}{\rho}\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}\rho)\right\}\Bigg|_{\rho=1} &= -c_3\frac{I_1E_1}{R_e^2}\left[(1-\nu)\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}) - \mathfrak{F}J_o(\sqrt{\mathfrak{F}})\right] \end{aligned} \tag{24b}$$

where we have utilized the derivatives of J_o as well as relation (18). If we equate equations (24a) and (24b) we have

$$\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}) - \kappa\frac{I_1E_1}{R_e^2}\left[(1-\nu)\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}) - \mathfrak{F}J_o(\sqrt{\mathfrak{F}})\right] = 0\,. \tag{25a}$$

Upon substitution of κ from (6) we obtain the nonlinear equation

$$\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}) - \frac{1}{4}\frac{\tilde{b}^3}{\delta^3}\frac{1}{\nu_o}\sqrt{\frac{\delta}{R_e}}\frac{\cos 2h\beta + \cosh 2h\beta + 2}{\sinh 2h\beta - \sin 2h\beta}\left[(1-\nu)\sqrt{\mathfrak{F}}J_1(\sqrt{\mathfrak{F}}) - \mathfrak{F}J_o(\sqrt{\mathfrak{F}})\right] = 0 \tag{25b}$$

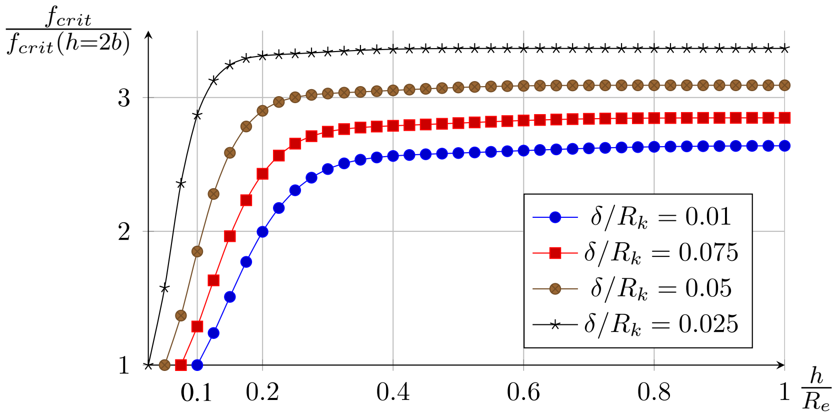
which provides the critical $\mathfrak{F}$. After solving this equation we can calculate the critical f_o or f_{crit} for the first loading case by using equations (12) and (6):

$$(1-\nu)\frac{R_e}{2b}\frac{f}{E} = \alpha(f_o - f) \quad \mapsto \quad f_o = f\left(1 + \frac{1-\nu}{\alpha}\frac{R_e}{2b}\frac{1}{E}\right)\,. \tag{26}$$

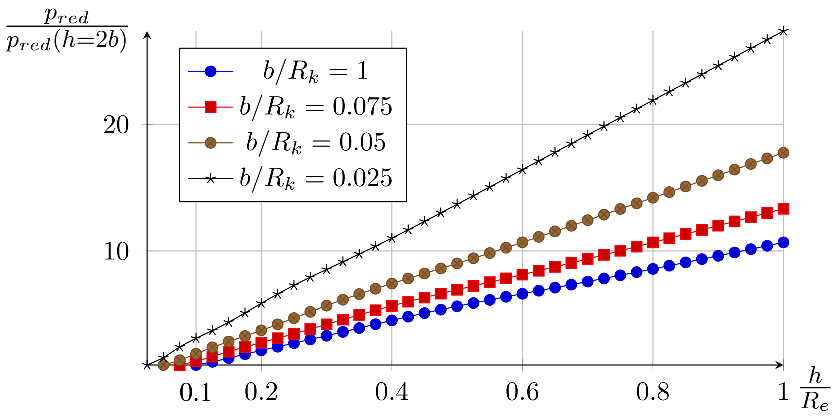
For the radial load p a similar line of thought provides the critical value:

$$p = f\left(\alpha + K\frac{R_k}{2bE}\right)\delta E/R_e^2 \tag{27}$$

A program has been written in the Fortran 90 language to solve the non-linear equation, moreover to calculate f_{crit}. We have assumed that (a) $\delta = 2b$; (b) the material of the shell and the plate is the same steel: $E = 2.1\times 10^5$ N/mm^2, $\nu = 0.3$; (c) if there is no shell the critical load is denoted by $f_{crit}(h = 2b)$. Under these conditions the quotient $f_{crit}/f_{crit}(h = 2b)$ depends on the quotients $2b/R_e = \delta/R_e$ and h/R_e – the latter is measured on the horizontal axis:



If the shell is subjected to the distributed load p we denote the intensity of the equivalent force system acting on the circle with radius R_e by p_{red}. The diagram below shows the quotient $\frac{p_{red}}{p_{red}(h=2b)}$ against h/R_e.



6. SUMMARY

The paper presents the governing equations for the two parts of the structure including the boundary conditions for each part under the condition that the whole structure is divided into two parts. After clarifying the deformations both in the plate and in the shell we derive the non-linear equations that provide us the critical load for both loadings. A program has been developed and the computational results are presented in graphical formats. The results obtained prove that the height of the plate does not change the critical load if the height is larger than a certain value.

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