SEPARATUM

Acta Technica Academiae Scientiarum Hungaricae, Tomus 73 (3–4), pp. 363–399 (1972)

GEAR CALCULATION BY USING COMPLEX EXPRESSIONS

L. HUSZTHY*

[Manuscript received September 10, 1970]

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PUBLISHING HOUSE OF THE HUNGARIAN ACADEMY OF SCIENCES
VERLAG DER UNGARISCHEN AKADEMIE DER WISSENSCHAFTEN
MAISON D'EDITIONS DE L'ACADEMIE DES SCIENCES DE HONGRIE
ИЗДАТЕЛЬСТВО АКАДЕМИИ НАУК ВЕНГРИИ

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In this paper, some geometric and mechanical properties of pairs of spur gears, characterized by straight teeth and parallel axes, are treated. The main points are: definition of the mating profile of a given one; analysis of the line of action; analysis of the geometric and mechanical conditions of mating; investigation of the gear ratio modification motivated by some deviation of the interaxis; calculation of the relative tooth-sliding velocity; calculation of the momentaneous normal tooth-force. One of the mating profiles, usually the pinion profile, is considered as given; by this datum, the line of action is determined and so is the other mating profile. A remarkable feature of the calculation method as described below, besides the usage of complex expressions, is the derivation of the results from given functions by which the profile of the pinion is determined.

Introduction

Since the literary source [2] includes both the description of the pinion profile on a complex plane, and the derivation of the equation by which the line of action is determined, and contains also the formation of the mating profile in detail, here this will be summarized only briefly as follows (NB: as regards the definition of the mating profile, the method applied in the quoted literary source somewhat deviates from the one proposed in the present paper).

In what follows, in considering a pair of straight spur gears having parallel axes,

the centre distance a,
and the gear ratio i

are given values, and i is constant. In other words, the assumption is taken as granted according to which the division ratio of the centre distance as defined by the common normal of the mating profiles at the pitch point is a constant value.

1. Based on a given profile, determination of the mating profile

In Fig. 1, the system of planar co-ordinates is shown that serves as the basic principle of our calculation. The point O_1 being the origin of the system, coincides with the rotation centre of the pinion. Axis y passes through

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the two centres O_1 and O_2 of the pinion and of the gear, respectively; axis x is perpendicular to axis y. In the following, subscript 1 refers to the pinion and 2 refers to the gear.

According to the accepted convention, the real quantities are represented by axis x, and axis y represents the imaginary ones. On axis y, the unity equals j (with $j^2 = -1$).

Anyone of the profiles in question is described by means of complex numbers; this number is composed as usual of a real and of an imaginary part, both of which are represented by a real function. By conformly chosen parameters, the geometric pictures of the profiles of both the pinion and the gear are characterized by the origin O_1 and O_2 respectively, and by an end-point that delineates the respective profile curve.

Vector quantities which serve to describe the profiles of the pinion and the gear, respectively, are denoted by Z_1 and Z_2. Their component parts are X_1, X_2 and Y_1, Y_2, written as capitals in order to avoid any confusion with small letters, since z_1 and z_2 refer to the teeth numbers, x represents the addendum modification factor and y represents the modification of the centre distance.

Now, first we shall fix the tooth profile of the pinion at any arbitrary time, $t = 0$. This profile, considered as being in the initial position can be described by the complex expression:

$$Z_{01} = X_{01}(\varphi_1) + jY_{01}(\varphi_1). \tag{1}$$

Further, φ_1 represents a real parameter suitably chosen as an angle, or a distance, etc. The vector Z_{01} is characterized by its origin in O_1 and its end-point travelling along the profile curve when the parameter value φ_1 varies within a given range. Subscript $_0$ refers to the initial position.

For the sake of further consideration, the functions $X_{01}(\varphi_1)$, $Y_{01}(\varphi_1)$ and their first and second derivatives are all assumed as being continuous.

When considering the profile as rotating in a positive sense at an angular velocity ω, the complex function Z_{01} has to be multiplied by the quantity

$$Z_{f1} = \cos\omega_1 t + j\sin\omega_1 t.$$

In terms of geometry, referring to some given point of the profile being originally in the initial position, the vector Z_{01} revolves in a positive sense, through an angle $\omega_1 t$. The rotating profile is expressed as follows:

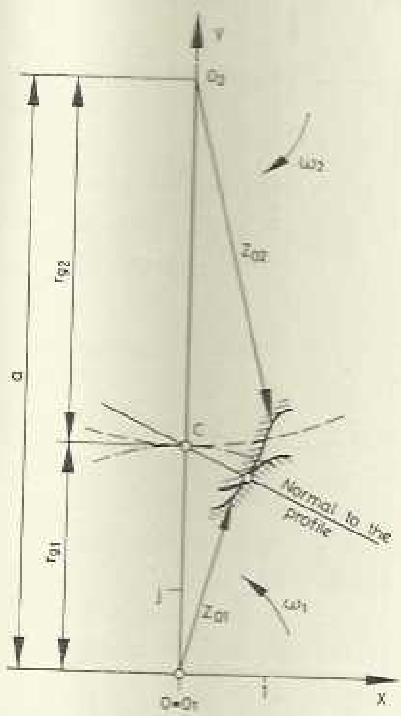
$$Z_1(\varphi_1; t) = [X_{01}(\varphi_1) + jY_{01}(\varphi_1)]\,[\cos\omega_1 t + j\sin\omega_1 t]$$

viz.

$$\begin{aligned} Z_1(\omega_1; t) = {} & [X_{01}(\varphi_1)\cos\omega_1 t - Y_{01}(\varphi_1)\sin\omega_1 t] + \\ & + j\,[X_{01}(\varphi_1)\sin\omega_1 t + Y_{01}(\varphi_1)\cos\omega_1 t]. \end{aligned} \tag{2}$$

In considering the function Z_1 and with the assumption that t is constant and φ_1 is variable, then the endpoint of vector Z_1 delineates the profile at the chosen moment. Again, taking φ_1 as constant and t as variable, the endpoint of Z_1 travels along the curve that corresponds to the path of the profile point determined by the chosen (initial) parameter value. Taking the tooth profile of pinion (1) as known, the profile of the gear (2) can be determined as follows:

The gear remains in a fixed position. The pinion's rolling circle r_{g1} rolls along the gear's rolling circle r_{g2}. By the family of curves represented as the pinion's profile positions during this rolling motion an envelope is generated that corresponds to the gear's tooth profile.



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364
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Fig. 1.

For the point of time $t = 0$, in the initial position, the pinion's tooth profile is characterized by equation (Fig. 2):

$$Z_{01} = Z_{01}(\varphi_1).$$

During this rolling motion, whereas the radius O_2O_1 revolves around centre O_2 through an angle ψ, O_1 finds its new position in O_1^*, and P^* is the new position of the initial P. In other terms, the arc $i \cdot r_{g1} \cdot \psi$ of the pinion performs a rolling motion along the arc $r_{g2}\psi$ of the gear. Herefrom, the angle of rotation of the pinion around its centre amounts to:

$$\psi + i\psi = (1 + i)\,\psi.$$

Of course, there exists a rigid connection between the tooth-profile and the rolling circle of the pinion. Accordingly, taking the vector Z_{01} (drawn from the origin O_1 to a certain point P) and, when the pinion performed the above-mentioned rolling motion, considering the vector Z_{01}^* (drawn from the new centre O_1^* to the point P^* in a new position), the angle included by these vectors also amounts to $(1 + i)\psi$.

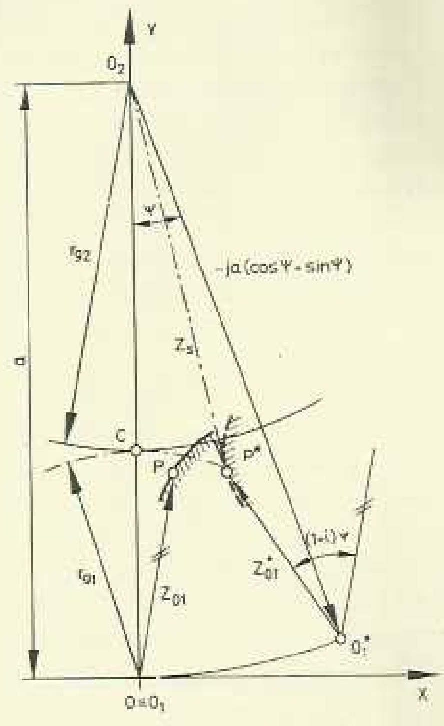

Fig. 2.

In order to find a more "natural" expression of the gear profile, we chose by simple shifting, the centre O_2 as the new origin of our system of co-ordinates, the direction of the axes being kept unchanged. Again, among the family curves we shall consider a curve co-ordinated to a certain rolling angle ψ, when the pinion performs the rolling motion along the rolling circle of the gear. Now, the local vector Z_s (drawn from O_2) of the point P^* that lies on the curve belonging to angle ψ, can be derived as follows:

We shall consider the vector $-ja$ that originates from O_2 and points to O_2. This vector will revolve through the angle ψ, i.e. it is multiplied by the rotating unity vector

$$\cos \psi + j \sin \psi;$$

so we obtain:

$$-ja (\cos \psi + j \sin \psi),$$

in which position the new vector, originating from O_2, points to O_1^*.

To this vector we add the other vector Z_{01}^*; this latter, originating from O_1^* and pointing to P^*, forms with vector Z_{01} an angle $(1 + i)\psi$. Accordingly

$$Z_{01}^* = Z_{01} [\cos (1 + i)\psi + j \sin (1 + i)\psi],$$

where the vector in square brackets represents the rotating unity vector.

Finally, we obtain:

$$\begin{aligned} Z_s &= -ja (\cos \psi + j \sin \psi) + Z_{01} [\cos (1 + i)\psi + j \sin (1 + i)\psi] = \\ &= -ja (\cos \psi + j \sin \psi) + [X_{01}(\varphi_1) + jY_{01}(\varphi_1)] \cdot \\ &\cdot [\cos (1 + i)\psi + j \sin (1 + i)\psi]. \end{aligned}$$

After the necessary arrangement:

$$\begin{aligned} Z_s(\varphi_1; \psi) = {} & [X_{01}(\varphi_1) \cos (1 + i)\psi - Y_{01}(\varphi_1) \sin (1 + i)\psi + a \sin \psi] + \\ & + j [X_{01}(\varphi_1) \sin (1 + i)\psi + Y_{01}(\varphi_1) \cos (1 + i)\psi - a \cos \psi]. \end{aligned} \tag{3}$$

Let us denote the real part (of vector Z_s) as X_s, and the imaginary part as Y_s. As we arrive from the profile point belonging to the parameter value φ_1 to the other point having the parameter value $(\varphi_1 + d\varphi_1)$, and the pinion rotates at an angle $d\psi$, we can observe that the total change of the vector Z_s is equal to

$$dZ_s = \left(\frac{\partial X_s}{\partial \varphi_1} + j \frac{\partial Y_s}{\partial \varphi_1} \right) d\varphi_1 + \left(\frac{\partial X_s}{\partial \psi} + j \frac{\partial Y_s}{\partial \psi} \right) d\psi. \tag{4}$$

The postulate is obvious that this vector should be parallel to the differential change $dZ_{b\varphi_1}$ viz. to a tangent of the vector $Z_b = Z_{02}(\varphi_1)$ which represents the desired envelope; this change is expressed by

$$dZ_{b\varphi_1} = \left(\frac{\partial X_b}{\partial \varphi_1} + j \frac{\partial Y_b}{\partial \varphi_1} \right) d\varphi_1 , \tag{5}$$

of course, every single point of the envelope coincides with a point lying on one curve belonging to the family. Thus, for all these points:

$$Z_b = Z_s ,$$

and

$$X_b = X_s ,$$
$$Y_b = Y_s . \tag{6}$$

In view of the parallelity between the vectors dZ_b an dZ_s, the quotient of them is a purely real quantity, viz. with an imaginary part equalling zero. Thus, with the equalities of (6) we obtain:

$$\mathrm{Im} \frac{\left(\frac{\partial X_s}{\partial \varphi_1} + j \frac{\partial Y_s}{\partial \varphi_1}\right) d\varphi_1 + \left(\frac{\partial X_s}{\partial \psi} + j \frac{\partial Y_s}{\partial \psi}\right) d\psi}{\left(\frac{\partial X_s}{\partial \varphi_1} + j \frac{\partial Y_s}{\partial \varphi_1}\right) d\varphi_1} = 0 .$$

After division and simplifying arrangement we obtain:

$$\left(\frac{\partial X_s}{\partial \varphi_1} \cdot \frac{\partial Y_s}{\partial \psi} - \frac{\partial X_s}{\partial \psi} \cdot \frac{\partial Y_s}{\partial \varphi_1}\right) d\varphi_1 \cdot d\psi = 0 ,$$

and in the form of a determinant:

$$D = \begin{vmatrix} \frac{\partial X_s}{\partial \varphi_1} & \frac{\partial X_s}{\partial \psi} \\ \frac{\partial Y_s}{\partial \varphi_1} & \frac{\partial Y_s}{\partial \psi} \end{vmatrix} = 0 . \tag{7}$$

The partial derivatives of X_s and Y_s with respect to φ_1 and ψ will be substituted in D (7). Since in the present examination the actual relationship between X_{01} and φ_1 (and between Y_{01} and φ_1 respectively) are not defined, the derivatives with respect to φ_1 are indicated by an upper comma; further, the denotation $(1 + i) = k$ is introduced. Thus we can write:

$$\begin{vmatrix} X'_{01} \cos k\psi - Y'_{01} \sin k\psi & -X_{01} k \sin k\psi - Y_{01} k \cos k\psi + a \cos \psi \\ X'_{01} \sin k\psi + Y'_{01} \cos k\psi & X_{01} k \cos k\psi - Y_{01} k \sin k\psi + a \sin \psi \end{vmatrix} = 0,$$

that gives (with $k - 1 = i$)

$$(i + 1)(X_{01} X'_{01} + Y_{01} Y'_{01}) - a (X'_{01} \sin i\psi + Y'_{01} \cos i\psi) = 0. \tag{8}$$

From this, parameter ψ will be expressed as a function of φ_1. By substituting the resulting value into the expression of Z_r, equation of the gear profile is obtained in a system of co-ordinates of axes parallel to the original axes and having the origin in O_2. This equation takes the form:

$$Z_2\,[\varphi_1; \psi(\varphi_1)] = Z_2(\varphi_1).$$

It is possible, that the actual expression is rather intricate; its main advantage consists in the fact that the gear profile is described as a function of φ_1, this latter being the parameter of the pinion-profile. In this way, to every single point of the gear profile, a special mating point of the pinion profile is co-ordinated. In some actual cases it is possible to modify the equation of the gear profile in such a way, by which a more suitable parameter can be established.

2. Equation of the line of action

The basis of this equation is given by the condition that the common normal of the mating profiles always passes through the point C as indicated in Fig. 3.

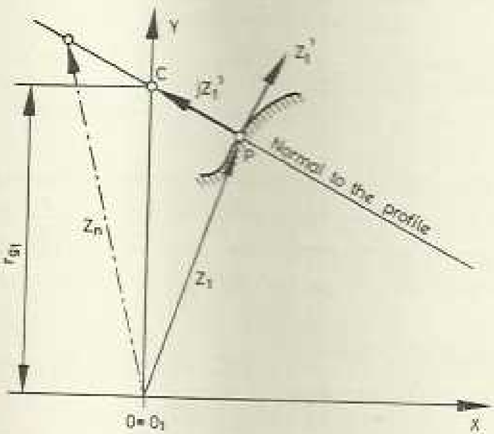

Fig. 3.

According to our consideration, point P is the new point of contact at the period of time t elapsed after the initial position. The local vector $Z_1(\varphi_1; t)$ refers to the actual mating point of the pinion's tooth profile. The derivative with respect to φ_1 shall usually be indicated by an upper comma; consequently, Z_1' denotes the tangent of the profile in point P, and jZ_1' denotes at the same point the perpendicular to the profile. Thus, the local vector Z_n

belonging to an arbitrary point lying on the normal of the profile at the momentaneous point of contact is expressed as follows:

$$Z_n(\varphi_1; t; \lambda) = Z_1(\varphi_1; t) + j\lambda Z_1'(\varphi_1; t);$$
$$(-\infty < \lambda < +\infty)$$

with λ as a real parameter. Point P is effectively a point of mating contact, if the above described normal to the profile passes through the mentioned point C. In other words, a triad of the parameter values (φ_1, t, λ) should exist for which

$$\begin{aligned} \text{Re} \ \ Z_n &= 0, \\ \text{Im} \ \ Z_n &= r_{g1} . \end{aligned} \tag{9}$$

By using Eq. (2) by which Z_1 is defined, the normal line of the profile is described by the equation:

$$\begin{aligned} Z_n = {} & [(X_{01} \cos \omega_1 t - Y_{01} \sin \omega_1 t) - \lambda(X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t)] + \\ & + j [(X_{01} \sin \omega_1 t + Y_{01} \cos \omega_1 t) + \lambda(X_{01}' \cos \omega_1 t - Y_{01}' \sin \omega_1 t)], \end{aligned}$$

from which the detailed conditions implied by equations (9) can be written:

$$\begin{aligned} (X_{01} \cos \omega_1 t - Y_{01} \sin \omega_1 t) - \lambda(X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t) &= 0 , \\ (X_{01} \sin \omega_1 t + Y_{01} \cos \omega_1 t) + \lambda(X_{01}' \cos \omega_1 t - Y_{01}' \sin \omega_1 t) &= r_{g1} . \end{aligned} \tag{10}$$

The above Eqs (10) describe the relationship between the parameter φ_1 belonging to a certain point of the pinion profile and the point of time t at which the profile point referred to represents a point of active mating.

The first equation according to (10) gives us the expression

$$\lambda = \frac{X_{01} \cos \omega_1 t - Y_{01} \sin \omega_1 t}{X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t} \ ; \ (X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t \neq 0) ,$$

with this, the second equation of (10) can be written as follows:

$$X_{01} \sin \omega_1 t + Y_{01} \cos \omega_1 t + \frac{X_{01} \cos \omega_1 t - Y_{01} \sin \omega_1 t}{X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t} (X_{01}' \cos \omega_1 t - Y_{01}' \sin \omega_1 t) = r_{g1} .$$

After elimination of the fractionel members and rearrangement

$$X_{01} X_{01}' + Y_{01} Y_{01}' - r_{g1} (X_{01}' \sin \omega_1 t + Y_{01}' \cos \omega_1 t) = 0,$$

with $r_{g_1} = a/(1 + i)$ we obtain:

$$(1 + i)(X_{01} X'_{01} + Y_{01} Y'_{01}) - a(X'_{01} \sin \omega_1 t + Y'_{01} \cos \omega_1 t) = 0 = \\ = V(\varphi_1; t). \tag{11}$$

This function $V(\varphi_1; t) = 0$ represents the interconnection between the parameter of an arbitrary point on the profile and the point of time t at which the point referred to forms the actual point of active mating.

With the assumption $\partial V/\partial t \neq 0$, by Eq. (11) we can obtain an explicit expression of the time t as a function of the parameter φ_1. When this function

$$t = t(\varphi_1)\ ,$$

is substituted into Eq. (2) of the rotating profile (1), the following equation of the line of action is obtained:

$$Z_t = Z_1[\varphi_1; t(\varphi_1)] = [X_{01}(\varphi_1) \cos \omega_1 t(\varphi_1) - Y_{01}(\varphi_1) \sin \omega_1 t(\varphi_1)] + \\ + j[X_{01}(\varphi_1) \sin \omega_1 t(\varphi_1) + Y_{01}(\varphi_1) \cos \omega_1 t(\varphi_1)]. \tag{12}$$

3. Conditions of mating

In the Eq. (11) of the function $V(\varphi_1; t) = 0$ the second member on the left-hand side can be transformed:

$$X'_{01} \sin \omega_1 t + Y'_{01} \cos \omega_1 t = \\ = \sqrt{X'^2_{01} + Y'^2_{01}} \left(\frac{X'_{01}}{\sqrt{X'^2_{01} + Y'^2_{01}}} \sin \omega_1 t + \frac{Y'_{01}}{\sqrt{X'^2_{01} + Y'^2_{01}}} \cos \omega_1 t \right),$$

and by using the denotations:

$$\frac{X'_{01}}{\sqrt{X'^2_{01} + Y'^2_{01}}} = \cos \delta,$$

$$\frac{Y'_{01}}{\sqrt{X'^2_{01} + Y'^2_{01}}} = \sin \delta,$$

we obtain:

$$X'_{01} \sin \omega_1 t + Y'_{01} \cos \omega_1 t = \sqrt{X'^2_{01} + Y'^2_{01}} \sin (\omega_1 t + \delta)$$

and

$$(1 + i)(X_{01} X'_{01} + Y_{01} Y'_{01}) - a \sqrt{X'^2_{01} + Y'^2_{01}} \sin (\omega_1 t + \delta) = 0\ ,$$

and finally:

$$t(\varphi_1) = \frac{1}{\omega_1}\left[\arc\sin \frac{(1+i)(X_{01}X'_{01}+Y_{01}Y'_{01})}{a\sqrt{X'^2_{01}+Y'^2_{01}}} - \arc\tan \frac{Y'_{01}}{X'_{01}}\right]. \quad (13)$$

$$(X'_{01} \neq 0)$$

Considering the contact of a pair of mating teeth, two conditions should be fulfilled:

a) every point of the profile must form a mating point once, and only once;

b) at any point of time, only one point of the profile must form a mating point.

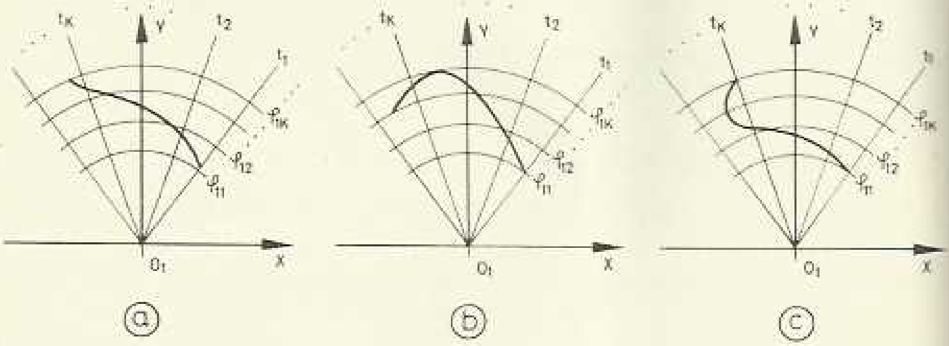

Fig. 4.

A certain type of a line of action is shown in Fig. 4*a* that fulfills the conditions *(a) (b)* given above.

In the type of a line of action, as delineated in Fig. 4*b*, there are profile points which, during the mating action, come twice into contact.

The case shown in Fig. 4*c* is characterized by a simultaneous mating contact of two points of a profile.

In order to avoid the above mentioned two possibilities, it is postulated that *the function* $t = t(\varphi_1)$ according to (13) must rigorously form *a monotonic function* during the mating period.

In Eq. (13) the absolute argumentum value of the function arc sin must have, of course, a maximum equalling 1:

$$\left|\frac{(1+i)(X_{01}X'_{01}+Y_{01}Y'_{01})}{a\sqrt{X'^2_{01}+Y'^2_{01}}}\right| \leq 1\,. \quad (14)$$

The geometric sense of this condition can be formulated as follows:
We establish the quotient with a vector Z_{01} as numerator drawn to an arbitrary

point of the profile, and with the tangent vector Z'_{01} (at the same point) as denominator:

$$\frac{Z_{01}}{Z'_{01}} = \frac{X_{01}+jY_{01}}{X'_{01}+jY'_{01}} = \frac{(X_{01}X'_{01}+Y_{01}Y'_{01})+j(X'_{01}Y_{01}-X_{01}Y'_{01})}{X'^2_{01}+Y'^2_{01}}.$$

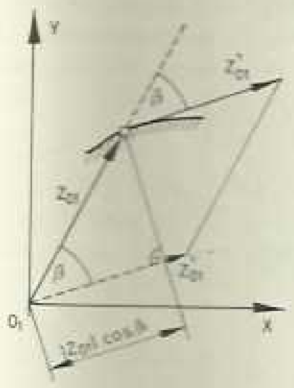

Fig. 5.

Let β be the sign of the angle formed by vector Z_{01} with vector Z'_{01} (Fig. 5), we shall find as the real part of the quotient defined above:

$$\mathrm{Re}\,\frac{Z_{01}}{Z'_{01}} = \mathrm{Re}\left[\frac{|Z_{01}|}{|Z'_{01}|}(\cos\beta + j\sin\beta)\right] = \frac{|Z_{01}|}{|Z'_{01}|}\cos\beta,$$

where

$$|Z_{01}|\cos\beta = |Z'_{01}|\,\mathrm{Re}\,\frac{Z_{01}}{Z'_{01}} =$$

$$= \sqrt{X'^2_{01}+Y'^2_{01}}\,\frac{X_{01}X'_{01}+Y_{01}Y'_{01}}{X'^2_{01}+Y'^2_{01}} = \frac{X_{01}X'_{01}+Y_{01}Y'_{01}}{\sqrt{X'^2_{01}+Y'^2_{01}}}$$

equals the length of the image of vector Z_{01} projected onto the direction of vector Z'_{01}.

In view of

$$\frac{1+i}{a} = \frac{1}{r_{g1}},$$

Eq. (14) takes the form:

$$\left|\frac{X_{01}X'_{01}+Y_{01}Y'_{01}}{\sqrt{X'^2_{01}+Y'^2_{01}}}\right| \le r_{g1}. \qquad (15)$$

According to the Eq. (15) we can state *that, considering a local vector co-ordinated to an arbitrary profile point, the length of its image projected onto the* *direction of the tangent at the same point equals, at maximum, the radius of the rolling circle.*

We shall find in Fig. 6 three different points of the pinion profile. Considering the point P_2, we can see that the length of the image of its local vector $Z_{01}(P_2)$ projected onto the direction of its tangent vector $Z'_{01}(P_2)$ exactly equals the radius r_{g1} of the rolling circle. This point P_2 may be a mating point, namely, when considering the profile normal at point P_2, we find that, by rotation of the pinion, its tangential point P' on the rolling circle falls so as to coincide with point C.

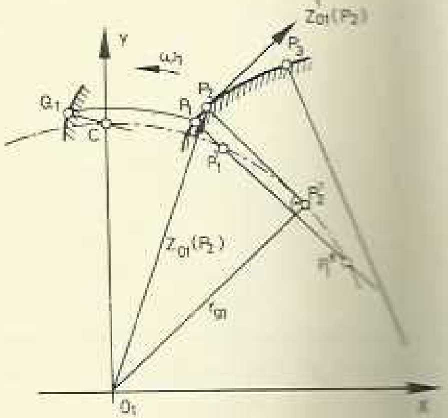

Fig. 6.

The profile normal at point P_1 intersects the rolling circle in two points P'_1 and P''_1. When, during rotation point P'_1 coincides with point C, the profile point P_1 comes into the mating position Q_1. After a further period of rotation, it is point P''_1 which coincides with the pitch point C, thus the original profile point P_1 again comes into mating position: this corresponds to the special case as shown in Fig. 4*b*: one special point of the profile comes twice into a mating position. Usually the mating period of the profile takes such a short time that there is, for such a profile point, no opportunity to come to mating a second time.

Again, profile point P_2 assumes a mating position only once.

Further, the normal to the profile at point P_3 has no intersection with the rolling circle and hence this point never assumes any mating position.

For the function (13) the postulate is cogent that

$$X'_{01} \neq 0, \tag{16}$$

otherwise the expression

$$\operatorname{arc\,tan} \frac{Y'_{01}}{X'_{01}}$$

becomes meaningless.

Since the expression

$$Z'_{01} = X'_{01} + j\, Y'_{01}$$

represents the directional vector of the tangent at some (arbitrary) point of the profile, the geometric sense of formula (16) can be conceived as follows: at the profile there does not exist any special point for which the equality $X'_{01} = 0$ might be true. Namely, in the opposite case, the profile function

$$Y = f(X)$$

would not be differentiated at this point, and the respective tangent line would be parallel to the Y axis. In terms of geometry, it is possible to find a tangent line that is parallel to the Y axis. In order to solve the problem of such a case, for which the equality

$$X^{*\prime}_{01}(\varphi^*_1) = 0$$

holds true, then the second member of function (13) would reach the limit $\pi/2$ as corresponding to the trend

$$\varphi_1 \to \varphi^*_1 .$$

For the formula (13) the inequality

$$X_{01}X'_{01} + Y_{01}Y'_{01} \neq 0 \tag{17}$$

incorporates a condition sine qua non.

Namely, when the equality

$$X_{01}X'_{01} + Y_{01}Y'_{01} = 0$$

would hold true, we had to set up

$$X_{01}X'_{01} = -Y_{01} \cdot Y'_{01} ,$$

or

$$\frac{Y'_{01}}{X'_{01}} = -\frac{X_{01}}{Y_{01}} = -\frac{1}{(Y_{01}/X_{01})} . \tag{18}$$

In plain words, Y'_{01}/X'_{01} represents the slope of the tangent line at the mating point of the profile, and Y_{01}/X_{01} represents the slope of the local vector at the same point. In other terms, the equality (18) should represent a position, in which the local vector of the mating point would be perpendicular to the tangent line at the same point (Fig. 7). In such a case, no normal force action can exist between the teeth (the limit case of friction).

To sum up: when the toothing of a pair of gears has to be designed, we are, in principle, free to choose the profile curve of one of the pairs: postulated is that the function (13) should be strongly monotonic, and that the relationships specified in Eqs (15), (16) and (17) should hold true.

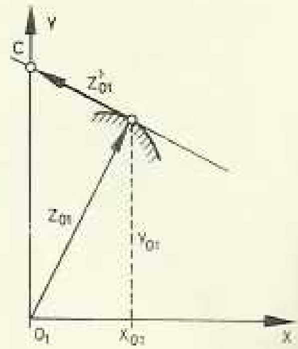

Fig. 7.

Of course, a monotonic function (13) involves the relationship, that with increasing values of the pinion profile parameter φ_1, the length of local vectors should also increase.

4. Sensitive alteration of the gear ratio to any variation of the centre distance

When the basic principle concerning the normal to the profile is maintained, then the gear ratio is constant during the mating action, if only the centre distance exactly equals the theoretic value calculated.

Now, some deviation of the centre distance may occur, due either to technological errors or faulty assemblage, or else to deformations caused by large forces. When some alteration of this distance happens to exist, generally, the common normal of the mating profiles does not pass through the pitch point (C); what is more, the intersection point of the normal to the profile with the central line O_1O_2 may move away during the mating of one pair of teeth; consequently, the gear ratio, first, does not equal the theoretic one, and secondly it varies during the mating period, too.

Taking into consideration the case when, from the kinematical view-point, the pair of gears have correct profiles, but in running the centre distance becomes altered, we are allowed to assume that a certain value of the centre distance is given, and to state that the formation of the profile curves does not conform with the basic theorem as regards the common normal. Thus, this problem can be put up by assuming a given centre distance and some given profiles that do not comply with the known principle underlying the common normal. These profile curves may be quite arbitrary except for the condition of a continuous mating contact.

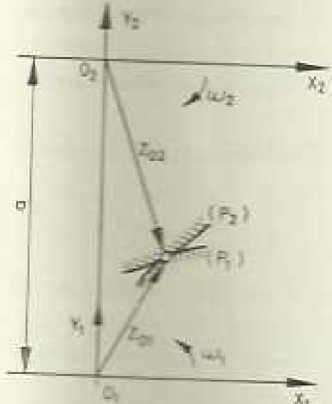

Fig. 8.

Let us establish, according to Fig. 8, the following equations: first

$$Z_{01} = X_{01}(\varphi_1) + jY_{01}(\varphi_1)$$

represents the local vector to the profile of pinion (1) expressed in the system of co-ordinates $x_1 y_1$ having the origin 0_1; secondly

$$Z_{02} = X_{02}(\varphi_2) + jY_{02}(\varphi_2)$$

represents the local vector to the profile of gear (2) in the system of co-ordinates x_2, y_2 having the origin 0_2.

Suppose that, in the initial position indicated by the index "0" the teeth of the two wheels are just in contact, and, of course, the value a of the centre distance is given.

When the pinion revolves, in a positive sense, at the angular speed

$$\omega_1 = 1,$$

so does the gear revolve in a negative sense, at an angular speed

$$\omega_2 = \frac{\omega_1}{i};$$

and with $\omega_1 = 1$ we have:

$$\omega_2 = \frac{1}{i} = \omega_2(t)$$

where, in general, i is a function of time.

In a period of time t the pinion revolves through the angle

$$\omega_1 \cdot t = 1 \cdot t = t$$

and the revolving angle of the gear is, at the same time:

$$T = \int_0^t \omega_2(t)\, dt \, .$$

We shall find the equation of the profiles revolving at the corresp angular velocity by multiplying the respective profile vector with th responding rotating unity vector. Thus, we can write the for pinion (1)

$$Z_1 = Z_{01} \cdot Z_{f1} \, ,$$

where

$$Z_{f1} = \cos t + j \sin t$$

is the unity vector rotating in a positive sense at the angular velo (around the centre 0_1);

and for the gear $\qquad Z_2 = Z_{02} \cdot Z_{f2} \, ,$

where

$$Z_{f2} = \cos T - j \sin T$$

is the unity vector rotating throug angle T in a negative sense (arou centre O_2).

Thus, the revolving profiles are expressed as follows:

$$Z_1 = Z_{01} Z_{f1} = (X_{01} + jY_{01})\,(\cos t + j \sin t),$$

$$Z_1 = (X_{01} \cos t - Y_{01} \sin t) + j(X_{01} \sin t + Y_{01} \cos t),$$

and

$$Z_2 = Z_{02}\, Z_{f2} = (X_{02} + jY_{02})\,(\cos T - j \sin T),$$

$$Z_2 = (X_{02} \cos T + Y_{02} \sin T) + j(-X_{02} \sin T + Y_{02} \cos T).$$

Considering the range of mating action characterized by $t_1 \leq t \leq$ postulate must be fulfilled that the two profiles always contact at a point in which the respective tangent lines are in coincidence. In th the following equations can be written as:

$$Z_1 = ja + Z_2,$$
$$Z_1' = \mu(ja + Z_2)' = \mu Z_2'. \tag{21}$$

(Symbols with an upper comma represent the corresponding first derivative with respect to φ_1 and φ_2, respectively; μ denotes a factor of proportionality.) After seperating into component parts we obtain:

$$\begin{aligned}
X_{01} \cos t - Y_{01} \sin t &= X_{02} \cos T + Y_{02} \sin T, \\
X_{01} \sin t + Y_{01} \cos t &= a - X_{02} \sin T + Y_{02} \cos T, \\
X_{01}' \cos t - Y_{01}' \cos t &= \mu(X_{02}' \cos T + Y_{02}' \sin T), \\
X_{01}' \sin t + Y_{01}' \cos t &= \mu(-X_{02}' \sin T + Y_{02}' \cos T).
\end{aligned} \tag{22}$$

The 4th equation of (22) will be divided by the 3rd:

$$\frac{X_{01}' \sin t + Y_{01}' \cos t}{X_{01}' \cos t - Y_{01}' \sin t} = \frac{-X_{02}' \sin T + Y_{02}' \cos T}{X_{02}' \cos T + Y_{02}' \sin T},$$

with

$$X_{01}' \cos t - Y_{01}' \sin t \neq 0$$

and

$$X_{02}' \cos T + Y_{02}' \sin T \neq 0.$$

There follows the operation of simultaneous division, namely, numerator and denominator at the left side will be divided by $\cos t$, and the same has to be done by $\cos T$ at the right side, where $\cos t \neq 0$, and $\cos T \neq 0$:

$$\frac{X_{01}' \tan t + Y_{01}'}{X_{01}' - Y_{01}' \tan t} = \frac{-X_{02}' \tan T + Y_{02}'}{X_{02}' + Y_{02}' \tan T},$$

and rearranged:

$$\begin{aligned}
& X_{01}' X_{02}' \tan t + X_{01}' Y_{02}' \tan t \cdot \tan T + X_{02}' Y_{01}' + Y_{01}' Y_{02}' \tan T = \\
& = -X_{01}' X_{02}' \tan T + X_{01}' Y_{02}' + X_{02}' Y_{01}' \tan t \tan T - Y_{01}' Y_{02}' \tan t;
\end{aligned}$$

viz.

$$\begin{aligned}
& \tan T [\tan t \, (X_{01}' Y_{02}' - X_{02}' Y_{01}') + X_{01}' X_{02}' + Y_{01}' Y_{02}'] = \\
& = X_{01}' X_{02}' - X_{02}' Y_{01}' - \tan t \, (X_{01}' X_{02}' + Y_{01}' Y_{02}'),
\end{aligned}$$

where from:

$$\tan T = \frac{(X_{01}' Y_{02}' - X_{02}' Y_{01}') - \tan t \, (X_{01}' X_{02}' + Y_{01}' Y_{02}')}{\tan t \, (X_{01}' Y_{02}' - X_{02}' Y_{01}') + (X_{01}' X_{02}' + Y_{01}' Y_{02}')}.$$

By uniting the system of equations (22) and the above formula for (tan T) we obtain:

$$X_{01}\cos t - Y_{01}\sin t = X_{02}\cos T + Y_{02}\sin T,$$

$$X_{01}\sin t + Y_{01}\cos t = a - X_{02}\sin T + Y_{02}\cos T, \tag{23}$$

$$\tan T = \frac{(X'_{01}Y'_{02} - X'_{02}Y'_{01}) - (X'_{01}X'_{02} + Y'_{01}Y'_{02})\tan t}{(X'_{01}Y'_{02} - X'_{02}Y'_{01})\tan t + (X'_{01}X'_{02} + Y'_{01}Y'_{02})},$$

a system that gives a solution for the problem in question.

When the parameters φ_1 and φ_2 are cancelled, and by due rearrangement, the quantity T will appear as a function of t, then the gear ratio $i(t)$ can already be expressed in an explicit form, namely:

$$T = \int_0^t \omega_2(t)\,dt,$$

$$\frac{d}{dt}\int_0^t \omega_2(t) = T'(t),$$

$$\omega_2(t) = T'(t),$$

and finally

$$i = \frac{\omega_1}{\omega_2} = \frac{1}{T'(t)}.$$

Of course, the nature of the profile function is decisive as far as the solution of the equations may become difficult or easy. In some given case the method for finding a solution is highly intricate. Cases may occur where the transcendental expressions can not be solved in a closed form. An issue is possible by finding some approximation, generally, by applying some iterative method of estimating, often by means of an electronic computer.

Considering the methods known at present, the difficulty can be described as follows: By means of suitably chosen parameters, and for profile curves in contact at the initial position, generally, with a theoretically correct centre distance, the respective equation is rather simple.

In the case of a varied centre distance (e.g. when it increases), one of the gears has to be rotated through a very small angle in order to create a new mating position. For further calculation, such a new configuration represents the initial position. In such a position, the new equation that describes the profile curve, after revolution through a small angle, may assume a rather complicated form.

Our calculation can be simplified when we rely on the usual equation co-ordinated to the initial position with an exact theoretic centre distance.

In this way we shall investigate angle λ_1 (for the pinion) and angle λ_2 (for the gear) through which the respective revolution should be performed for the purpose that equations (21) relating to the common profile point and to the parallelity of the tangent lines should hold true.

For the calculation of the gear ratio i the system (23) of equations is quite adequate; only instead of term t it is angle λ_1, and instead of T it is angle λ_2 that should be substituted in all three equations of system (23). In concreto, angle value λ_1 of the pinion revolution is taken as a given datum, by means of which the parameter values φ_1 and φ_2, and the angle λ_2 have to be calculated.

When once the gear revolution angle λ_2 is obtained, it is obvious that in this position the profile points defined by φ_1 and φ_2 are in mating contact. With the values λ_1 and φ_1, the equation of the normal to the profile can be set down, this line being the common normal. Then the momentaneous pitch point C'' is found as the intersection point of the said normal and the central line O_1O_2.

The quotient

$$\frac{\overline{C''O_2}}{\overline{C''O_1}} \tag{24}$$

equals the momentaneous value of the gear ratio. By iterating this calculation for various values of λ_1, a series of pairs of co-ordinated values λ_1 and i are obtained.

Besides, this method of calculation is well suited for finding the gear ratio of a cam type gearing, too, when the cam profile is a given curve.

5. The relative sliding velocity of teeth

In order to get the value of the relative tooth' sliding velocity, the actual peripheral velocity values generated by the revolution around the centres O_1 and O_2 should be calculated. Inasmuch as the profile curves comply with the condition as postulated for the common normal, the images of the respective vectors projected onto the common normal are equal to each other. In this sense, the vectorial difference of the two peripheral velocities is parallel to the common tangent line and already equals the relative sliding velocity (Fig. 9). Let us suppose a positive angular velocity ω_1 around the centre O_1 for the pinion, and of course, a resulting negative angular velocity $\omega_2 = \omega_1/i$ around the centre O_2 for the gear. Let point P be the locus of the mating contact. To this point, vector Z_1 drawn from origin O_1, and vector Z_2 drawn from origin O_2 are co-ordinated. Thus, we obtain as the value of the peripheral velocity $|v_1|$ in point P around centre O_1

$$|v_1| = |Z_1|\,\omega_1 \text{ (with } v_1 \text{ as a complex number);}$$

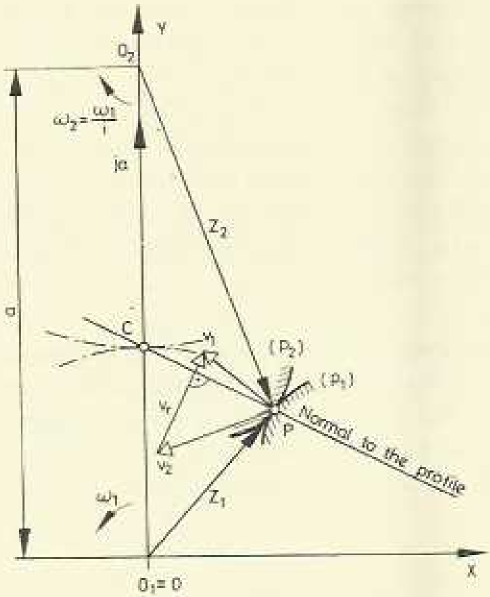

Fig. 9

v_1, being perpendicular to Z_1, represents the velocity vector in this form:

$$v_1 = |Z_1|\, \omega_1 \cdot j \frac{Z_1}{|Z_1|},$$

when $j\, Z_1/|\, Z_1\,|$ represents the unity vector which is perpendicular to vector Z_1. Consequently,

$$v_1 = j\omega_1 Z_1 . \tag{25}$$

In much the same way (for point P revolving around centre O_2)

$$|\, v_2\,| = |Z_2|\, \omega_2 = |Z_2| \frac{\omega_1}{i},$$

this v_2 being a complex number, too.

Now, v_2 being perpendicular to vector Z_2 at a direction determined by a revolution through (-90°), we obtain:

$$v_2 = |Z_2| \frac{\omega_1}{i} \left(-j \frac{Z_2}{|Z_2|} \right),$$

$$v_2 = -j \frac{\omega_1}{i} Z_2 . \tag{26}$$

At every mating point we have:

$$ja + Z_2 = Z_1,$$

or

$$Z_2 = Z_1 - ja,$$

that will be substituted in to (26):

$$v_2 = -j\frac{\omega_1}{i}(Z_1 - ja) = -\frac{a\omega_1}{i} - j\frac{\omega_1}{i}Z_1. \tag{27}$$

The relative sliding velocity v_r is obtained as the difference between vectors (25) and (27):

$$v_r = v_1 - v_2 = j\omega_1 Z_1 + \frac{a\omega_1}{i} + j\frac{\omega_1}{i}Z_1 = \frac{a\omega_1}{i} + j\left(\omega_1 Z_1 + \frac{\omega_1}{i}Z_1\right),$$

or

$$v_r = \omega_1\left(\frac{a}{i} + j\frac{1+i}{i}Z_1\right). \tag{28}$$

Of course, Z_1 is co-ordinated to the line of action: in other words, for vector $Z_1(\varphi_1; t)$ the term $t = t(\varphi_1)$ is understood in accordance with equation (13).

For common practice, in general, the magnitude of the relative sliding velocity is required. We calculate the absolute value of vector v_r as defined by formula (28) as follows:

since

$$Z_1 = [X_{01}\cos\omega_1 t - Y_{01}\sin\omega_1 t] + \\ + j[X_{01}\sin\omega_1 t + Y_{01}\cos\omega_1 t],$$

therefore

$$v_r = \left\{\left[\frac{a}{i} - \frac{1+i}{i}(X_{01}\sin\omega_1 t + Y_{01}\cos\omega_1 t)\right] + \\ + j\left[\frac{1+i}{i}(X_{01}\cos\omega_1 t - Y_{01}\sin\omega_1 t)\right]\right\}\omega_1,$$

and

$$|v_r|^2 = \frac{\omega_1^2}{i^2}\left[a^2 - 2a(1+i)(X_{01}\sin\omega_1 t + Y_{01}\cos\omega_1 t) + (1+i)^2(X_{01}^2 + Y_{01}^2)\right],$$

or

$$|v_r| = \frac{\omega_1}{i}\sqrt{a^2 - 2a(1+i)(X_{01}\sin\omega_1 t + Y_{01}\cos\omega_1 t) + (1+i)^2(X_{01}^2 + Y_{01}^2)} \tag{29}$$

and $t = t(\varphi_1)$ according to Eq. (13).

In some papers (e.g. [3] and [4]) dealing with frictional losses of gears we find the statement according to which it can be inferred from experiments with circular discs, that the friction ratio is a function of disc radii and velocity values. When these results should be taken into consideration in dealing with toothed gears, the friction losses can only be found when the various radii at the mating periods of the profile curvatures are taken into account. Now, with a given centre distance a and a given gear ratio i, the gear profile is univocally defined by the assumed pinion profile; consequently, with already known values of the angular velocity, the relative slip velocity depends on the geometric formation of the profiles — thus, a certain univocal relationship between the relative sliding velocity and the curvature radius at the mating point is an irrefutable consequence.

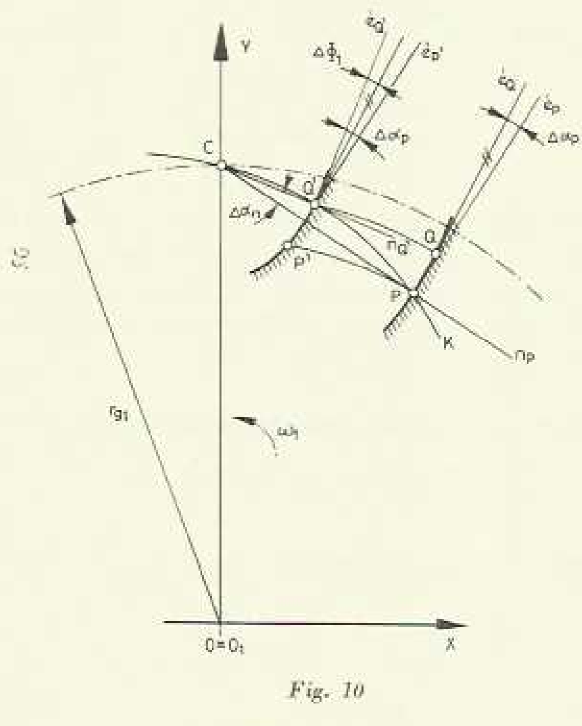

Fig. 10

Let the centre O_1 of the pinion, the centre distance a and the gear ratio i be given. By these data, the form of the line of action (k) is defined (Fig. 10).

Let point P be the contact point of the two profiles at a point of time t; at the same time, the tangent line ($\acute{e}_p$) at point P is perpendicular to the common normal (n_p) at the pitch point. When the elementary period of time Δt has elapsed, it is in point Q where the mating contact takes place. During this, in consequence of a revolution through angle $\Delta\Phi_1$, point P moves to P'; and Q moves to Q'.

The tangent line $(\acute{e}_{Q'})$ at point Q' is perpendicular to the profile normal $(n_{Q'})$. The angle between the two normals (n_p and $n_{Q'}$) will be denoted as $\Delta\alpha_n$.

The angle between the tangent lines at P and Q, as well as that between the tangent lines at P' an Q' is invariably $\Delta\alpha_p$.

Since at the initial position tangent $(\acute{e}_p)$ is perpendicular to the normal (n_p), and after a time Δt elapsed the tangent $(\acute{e}_{Q'})$ will be perpendicular to the normal $(n_{Q'})$, so we have:

$$\sphericalangle\,(\acute{e}_p; \acute{e}_{Q'}) = \Delta\alpha_n . \tag{30}$$

On the other hand, the angle between $(\acute{e}_p)$ and $(\acute{e}_{Q'})$ can be conceived as the sum of $\Delta\alpha_p$ and $\Delta\Phi_1$ because $\Delta\alpha_p$ is the angle between the tangents at P and Q in the initial position, and the tangent $(\acute{e}_Q)$ reaches the position $(\acute{e}_{Q'})$ after a revolution through the angle $\Delta\Phi_1$; consequently

$$\sphericalangle\,(\acute{e}_p; \acute{e}_{Q'}) = \Delta\alpha_p + \Delta\Phi_1 . \tag{31}$$

From (30) and (31) we have

$$\begin{aligned} \Delta\alpha_n &= \Delta\alpha_p + \Delta\Phi_1 , \\ \Delta\alpha_p &= \Delta\alpha_n - \Delta\Phi_1 . \end{aligned} \tag{32}$$

We denote the length of the profile arc $\widehat{PQ}$ as Δs_1. The mating point moves, during the time Δt along the arc Δs_1; thus, the average velocity of the mating point between P and Q amounts to

$$v_{1ae} = \frac{\Delta s_1}{\Delta t} .$$

Dividing both numerator and denominator by $\Delta\alpha_p$:

$$v_{1ae} = \frac{\Delta s_1/\Delta\alpha_p}{\Delta t/\Delta\alpha_p} = \frac{\Delta s_1}{\Delta\alpha_p} \cdot \frac{\Delta\alpha_p}{\Delta t} = \frac{\Delta s_1}{\Delta\alpha_p} \cdot \frac{\Delta\alpha_n - \Delta\Phi_1}{\Delta t} = \frac{\Delta s_1}{\Delta\alpha_p}\left(\frac{\Delta\alpha_n}{\Delta t} - \frac{\Delta\Phi_1}{\Delta t}\right). \tag{33}$$

Assuming a trend to the limit $\Delta t \to 0$ the angle $\Delta\alpha_p$ will similarly trend to $\Delta\alpha_p \to 0$.

In this sense

$$\lim_{\Delta\alpha_p \to 0}\left(\frac{\Delta_{s1}}{\Delta\alpha_p}\right) = \lim_{\Delta\alpha_p \to 0} \frac{1}{\Delta\alpha_p/\Delta s_1} = \frac{1}{g_1}$$

where, in conformity with its definition, g_1 denotes the curvature of the profile at point P.

Thus, the angular velocity of the normal to the profile revolving around point C is found to be:

$$\lim_{\Delta t \to 0} \frac{\Delta \alpha_n}{\Delta t} = \omega_n ,$$

and the angular velocity of the pinion's

$$\lim_{\Delta t \to 0} \frac{\Delta \Phi_1}{\Delta t} = \omega_1 .$$

Since the average velocity v_{rav} [at the left-hand side of Eq. (33)] assumes, when the limit $\Delta t \to 0$ is reached, the value v_{1t} which represents the momentaneous tangential velocity of the pinion in point P, we obtain:

$$v_{1t} = \frac{1}{g_1}(\omega_n - \omega_1).$$

Instead of the reciprocal of curvature, the radius of curvature (ϱ_1 in point P) can be substituted, namely:

$$v_{1t} = \varrho_1(\omega_n - \omega_1). \tag{34}$$

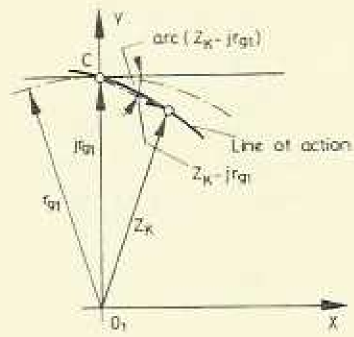

Fig. 11

According to a similar reasoning the tangential velocity of the gear at the mating point is found to be equal to:

$$v_{2t} = \varrho_2(\omega_n - \omega_2) \tag{35}$$

where ϱ_2 is the radius of curvature of the gear profile at P.

Thus, we find the relative sliding velocity to be expressed as:

$$v_r = | v_{1t} - v_{2t} | = | \varrho_1(\omega_n - \omega_1) - \varrho_2(\omega_n - \omega_2) |,$$

and reaaranged:

$$v_r = | (\varrho_1 - \varrho_2)\omega_n - (\varrho_1 \omega_1 - \varrho_2 \omega_2) |. \tag{36}$$

For this expression, the angular velocity ω_n will be calculated as follows:

The direction of a profile normal co-ordinated to an arbitrary mating point is defined by the vector (Fig. 11)

$$Z_k - jr_{g1}.$$

Taking into consideration the angle between this vector and the axis x, and forming the first derivative of this arc with respect to time, we obtain the angular velocity in question:

$$\omega_n = \frac{d}{dt} \operatorname{arc}(Z_k - jr_{g1}).$$

The vector Z_k, which describes the line of action, is a function of the pinion-parameter φ_1. According to the chain rule we have:

$$\omega_n = \frac{d}{d\varphi_1} \operatorname{arc}(Z_k - jr_{g1}) \frac{d\varphi_1}{dt}. \tag{37}$$

By using the known function $Z_k(\varphi_1)$ and by making the following transformation

$$\operatorname{arc}(Z_k - jr_{g1}) = \operatorname{arc}(X_k + jY_k - jr_{g1}) = \operatorname{arc}[X_k + j(Y_k - r_{g1})] =$$
$$= \operatorname{arc}\tan \frac{Y_k - r_{g1}}{X_k},$$

the derivative of the latter with respect to φ_1 can be directly deduced, the differential quotient $d\varphi_1/dt$ can be calculated from the function $V(\varphi_1; t) = 0$ according to Eq. (11).

6. The tooth force in normal direction

Suppose that — by neglecting the frictional force — the pinion is rotated by a constant moment M_1: Contact point of the profiles is the point P on the line of action (Fig. 12). Then, the tooth force acting in the normal direction is expressed:

$$F_n = \frac{M_1}{k}$$

with k being the lever arm of the force F_n. This k is found, when, considering the local vector co-ordinated to the mating point P, we draw its projection onto the direction of the tangent vector ${}_kZ_1'$ co-ordinated to point P. (The left-hand index indicates the co-ordination of this vector Z_1' to the line of action.)

In conformity with the geometric explanation of Eq. (14), we have:

$$k = \frac{X_1 X_1' + Y_1 Y_1'}{\sqrt{X_1'^2 + Y_1'^2}}$$
$$t = t(\varphi_1). \tag{38}$$

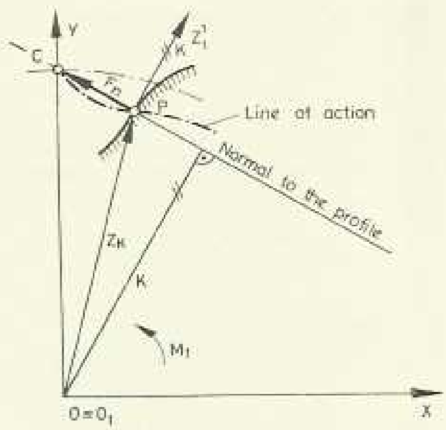

Fig. 12

Applying Eq. (2) by which the vector $Z_1 = Z_1(\varphi_1; t)$ is described, the numerator in (38) can be expressed as follows:

$$
\begin{aligned}
X_1 X_1' + Y_1 Y_1' = \\
&= [X_{01}(\varphi_1) \cos \omega_1 t(\varphi_1) - Y_{01}(\varphi_1) \sin \omega_1 t(\varphi_1)] \cdot \\
&\quad \cdot [X_{01}'(\varphi_1) \cos \omega_1 t(\varphi_1) - Y_{01}'(\varphi_1) \sin \omega_1 t(\varphi_1)] + \\
&+ [X_{01}(\varphi_1) \sin \omega_1 t(\varphi_1) + Y_{01}(\varphi_1) \cos \omega_1 t(\varphi_1)] \cdot \\
&\quad \cdot [X_{01}'(\varphi_1) \sin \omega_1 t(\varphi_1) + Y_{01}'(\varphi_1) \cos \omega_1 t(\varphi_1)] = \\
&= X_{01}(\varphi_1)\, X_{01}'(\varphi_1) + Y_{01}(\varphi_1)\, Y_{01}'(\varphi_1) .
\end{aligned}
$$

The denominator is expressed:

$$\sqrt{X_{01}'^2(\varphi_1) + Y_{01}'^2(\varphi_1)} .$$

As can be seen, neither the numerator nor the denominator of k is a function of t. Without referring to the dependence on φ_1 we can write:

$$k = \frac{X_{01} X_{01}' + Y_{01} Y_{01}'}{\sqrt{X_{01}'^2 + Y_{01}'^2}} . \tag{39}$$

Should the numerator equal zero, the rotating moment would vanish. Again, according to Eq. (17), the numerator cannot be equal to zero, neither can the denominator be equal to zero [in compliance with Eq. (16)].

Considering the profiles in question, the quantity k can never be equal to zero. Thus, with Eq. (39), the normal tooth force is expressed by:

$$F_n = M_1 \frac{\sqrt{X_{01}'^2 + Y_{01}'^2}}{X_{01}\, X_{01}' + Y_{01}\, Y_{01}'} . \tag{40}$$

7. Applications

We shall demonstrate, in point 7.1, a certain application of the relations as shown in chapters 1 . . . 6 especially for the case of a pair of gears consisting of a toothed wheel having cycloidal profiles and of another gear having rectilinear profiles. This special example can be taken as being useful because the results as well as the manner of their deduction by means of elementary notions are well known.

In point 7.2, a special case is dealt with.

7.1 *Pair of gears having a rectilinear and a cycloidal toothing, respectively*

With reference to Fig. 13, the toothed wheel (1) has epicycloidal profiles. This profile is generated by a circle having a radius c and revolving along the pitch circle. In the initial position of the profile, the revolving circle touches the pitch circle of gear (1) at the pitch point C. Now, the angle between the axis y and the line connecting centre O_1 with the centre of the rolling

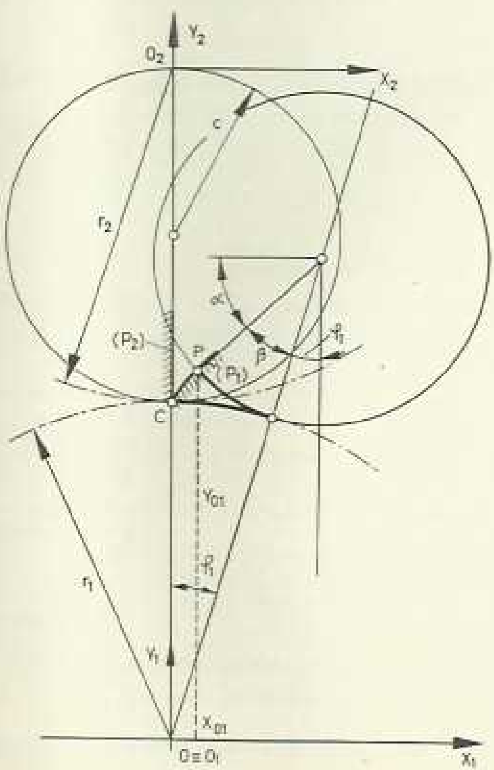

Fig. 13

circle should be chosen as the desired parameter φ_1. When the rolling circle rotates around its centre through the angle β, we have, with a pure rolling motion (without any sliding):

$$r_1 \varphi_1 = c\beta,$$

$$\beta = \frac{r_1}{c} \varphi_1. \tag{41}$$

The profile is delineated by a point of the rolling circle, the initial position of which lies at the pitch point C. In accordance with Fig. 13, the co-ordinates of the cycloidal point P are expressed:

$$X_{01} = (r_1 + c) \sin \varphi_1 - c \cos \alpha$$
$$Y_{01} = (r_1 + c) \cos \varphi_1 - c \sin \alpha.$$

In view of

$$\alpha = 90° - (\beta + \varphi_1),$$

or

$$\alpha + \beta + \varphi_1 = 90°,$$

we can write:

$$\begin{aligned} X_{01} &= (r_1 + c) \sin \varphi_1 - c \cos [90° - (\beta + \varphi_1)] = \\ &= (r_1 + c) \sin \varphi_1 - c \sin (\beta + \varphi_1) \\ Y_{01} &= (r_1 + c) \cos \varphi_1 - c \sin [90° - (\beta + \varphi_1)] = \\ &= (r_1 + c) \cos \varphi_1 - c \cos (\beta + \varphi_1) \end{aligned}$$

and combined with Eq. (41) we obtain:

$$\begin{aligned} X_{01} &= (r_1 + c) \sin \varphi_1 - c \sin \left(\frac{r_1}{c} + 1 \right) \varphi_1, \\ Y_{01} &= (r_1 + c) \cos \varphi_1 - c \cos \left(\frac{r_1}{c} + 1 \right) \varphi_1. \end{aligned} \tag{42}$$

For our numeric example we shall count with the concrete values:

$$r_1 = r_2 = 2, \quad c = 1$$

from where

$$a = 4, \quad i = 1.$$

With these data, we can write:

$$\begin{aligned} X_{01} &= 3 \sin \varphi_1 - \sin 3\varphi_1, \\ Y_{01} &= 3 \cos \varphi_1 - \cos 3\varphi_1, \end{aligned} \tag{43}$$

and for the profile (p_1) in initial position, with reference to Eq. (3) we obtain:

$$Z_{01} = (3 \sin \varphi_1 - \sin 3\varphi_1) + j(3 \cos \varphi_1 - \cos 3\varphi_1). \tag{7/1}$$

Let the gear (1) rotate in a positive sense at an angular velocity $\omega_1 = 1$, then the vector by which the rotating profile (p_1) is described can be expressed, with reference to Eq. (2), as follows:

$$Z_1 = [(3 \sin \varphi_1 - \sin 3\varphi_1) \cos t - (3 \cos \varphi_1 - \cos 3\varphi_1) \sin t] + \\ + j\, [(3 \sin \varphi_1 - \sin 3\varphi_1) \sin t + (3 \cos \varphi_1 - \cos 3\varphi_1) \cos t]. \tag{7/2}$$

In order to find the *formulae for profile* (2), Eq. (8) serves as basis; first, we will express the concrete formulae of the respective derivatives:

$$X'_{01} = 3 \cos \varphi_1 - 3 \cos 3\varphi_1 = 3(\cos \varphi_1 - \cos 3\varphi_1);$$

and

$$Y'_{01} = -3 \sin \varphi_1 + 3 \sin 3\varphi_1 = 3(-\sin \varphi_1 + \sin 3\varphi_1).$$

Thus, the Eq. (8) takes the form:

$$2\, [(3 \sin \varphi_1 - \sin 3\varphi_1) \cdot 3(\cos \varphi_1 - \cos 3\varphi_1) + (3 \cos \varphi_1 - \cos 3\varphi_1) \cdot 3(-\sin \varphi_1 + \sin 3\varphi_1)] - \\ - 4\, [3(\cos \varphi_1 - \cos 3\varphi) \sin \psi + 3(-\sin \varphi_1 + \sin 3\varphi_1) \cos \psi] = 0. \tag{44}$$

Now, in order to *express the variable* ψ as a function of φ_1, we first shall divide the expression (44) by the product $2 \cdot 3$; this results in the form:

$$(3 \sin \varphi_1 - \sin 3\varphi_1)(\cos \varphi_1 - \cos 3\varphi_1) + \\ + (3 \cos \varphi_1 - \cos 3\varphi_1)(-\sin \varphi_1 + \sin 3\varphi_1) = \\ = 2\, [(\cos \varphi_1 - \cos 3\varphi_1) \sin \psi + (-\sin \varphi_1 + \sin 3\varphi_1) \cos \psi]. \tag{44a}$$

After performing the operations at the *left-hand side* in (44a), we obtain the expression:

$$4 \sin \varphi_1 \cos \varphi_1 .$$

After division and multiplication of the right-hand side in (44a), by the following member:

$$\sqrt{(\cos \varphi_1 - \cos 3\varphi_1)^2 + (-\sin \varphi_1 + \sin 3\varphi_1)^2} = 2 \sin \varphi_1;$$

the right-hand side will take the form:

$$2 \cdot 2 \sin \varphi_1 \left[\frac{\cos \varphi_1 - \cos 3\varphi_1}{2 \sin \varphi_1} \sin \psi + \frac{\sin 3\varphi_1 - \sin \varphi_1}{2 \sin \varphi_1} \cos \psi \right] .$$

After transforming the members within the square brackets, by using the relation

$$\cos \varphi_1 - \cos 3\varphi_1 = 4 \cos \varphi_1 \sin^2 \varphi_1 ,$$

the first fraction can be written as follows:

$$\frac{4 \cos \varphi_1 \sin^2 \varphi_1}{2 \sin \varphi_1} = 2 \cos \varphi_1 \sin \varphi_1 = \sin 2\varphi_1 ,$$

and similarly, by using the relation

$$\sin 3\varphi_1 - \sin \varphi_1 = 2 \sin \varphi_1 \cos 2\varphi_1 ,$$

the second fraction is written:

$$\frac{2 \sin \varphi_1 \cos 2\varphi_1}{2 \sin \varphi_1} = \cos 2\varphi_1 .$$

With these simplified members, Eq. (44a) takes the form:

$$4 \sin \varphi_1 \cos \varphi_1 = 4 \sin \varphi_1 [\sin \psi \sin 2\varphi_1 + \cos \psi \cos 2\varphi_1].$$

and

$$\cos \varphi_1 = \cos (2\varphi_1 - \psi)$$

or

$$\varphi_1 = 2\varphi_1 - \psi,$$

viz.

$$\psi = \varphi_1 . \tag{45}$$

The expression of profile (p_2) is obtained, when the relation (45) is substituted into Eq. (3):

$$\begin{aligned} Z_2 [\varphi_1; \psi(\varphi_1)] &= Z_2(\varphi_1) = Z_{02} = \\ &= [(3 \sin \varphi_1 - \sin 3\varphi_1) \cos 2\varphi_1 - (3 \cos \varphi_1 - \cos 3\varphi_1) \sin 2\varphi_1 + 4 \sin \varphi_1] + \\ &+ j [(3 \sin \varphi_1 - \sin 3\varphi_1) \sin 2\varphi_1 + (3 \cos \varphi_1 - \cos 3\varphi_1) \cos 2\varphi_1 - 4 \cos \varphi_1]. \end{aligned}$$

The real part of vector Z_{02} can be written as:

$$\begin{aligned} X_{02} &= 3 \sin \varphi_1 \cos 2\varphi_1 - \sin 3\varphi_1 \cos 2\varphi_1 - 3 \cos \varphi_1 \sin 2\varphi_1 + \cos 3\varphi_1 \sin 2\varphi_1 + 4 \sin \varphi_1 = \\ &= -3 \sin \varphi_1 - \sin \varphi_1 + 4 \sin \varphi_1 = 0, \end{aligned}$$

and the imaginary part is:

$$\begin{aligned} Y_{02} &= 3 \sin \varphi_1 \sin 2\varphi_1 - \sin 3\varphi_1 \sin 2\varphi_1 + 3 \cos \varphi_1 \cos 2\varphi_1 - \cos 3\varphi_1 \cos 2\varphi_1 - 4 \cos \varphi_1 = \\ &= 3 \cos \varphi_1 - \cos \varphi_1 - 4 \cos \varphi_1 = -2 \cos \varphi_1 . \end{aligned}$$

Thus, the profile (p_2) is expressed in the system of co-ordinates (x_2; y_2) as follows:

$$\begin{aligned} X_{02} &= 0 \\ Y_{02} &= 2 \cos \varphi . \end{aligned} \tag{46}$$

This represents, in words, a straight line that coincides with the axis y. This result could be expected, undoubtedly, since the rolling circle has a radius $r = 1$, viz. half as large as the radius $r_2 = 2$ of the pitch circle within which the rolling motion takes place.

In this way the problem as set up in point 1 is solved.

We shall find the equation of the line of action. The concept of the function $V(\varphi_1; t)$ according to Eq. (11) is of a similar form as is the relation in Eq. (3) that served to define the envelope curve. The difference consists in substituting $\omega_1 t$ (as argument of the trigonometric functions) instead of $i\psi$. As far as the form of the line of action is concerned, the value of the angular velocity has no importance. For the sake of simplicity we assume $\omega_1 = 1$. When according to Eq. (8) the envelope curve was examined (with a gear ratio $i = 1$), we obtained

$$i\psi = \psi = \varphi_1$$

[see Eq. (45)], and in the present case, we find in an analogous way, that

$$\omega_1 t = t = \varphi_1 . \tag{47}$$

Putting this expression into Eq. (2) of profile (p_1), the vector that describes the line of action is found to be:

$$\begin{aligned} Z_k &= [(3 \sin \varphi_1 - \sin 3\varphi_1) \cos \varphi_1 - (3 \cos \varphi_1 - \cos 3\varphi_1) \sin \varphi_1] + \\ &+ j [(3 \sin \varphi_1 - \sin 3\varphi_1) \sin \varphi_1 + (3 \cos \varphi_1 - \cos 3\varphi_1) \cos \varphi_1]. \end{aligned} \tag{7/12}$$

The *real* part is expressed by:

$$\begin{aligned} X_k &= (3 \sin \varphi_1 - \sin 3\varphi_1) \cos \varphi_1 - (3 \cos \varphi_1 - \cos 3\varphi_1) \sin \varphi_1 = \\ &= -(\sin 3\varphi_1 \cos \varphi_1 - \cos 3\varphi_1 \sin \varphi_1) = -\sin (3\varphi_1 - \varphi_1) = -\sin 2\varphi_1 . \end{aligned}$$

and the *imaginary* part by:

$$Y_k = (3 \sin \varphi_1 - \sin 3\varphi_1) \sin \varphi_1 + (3 \cos \varphi_1 - \cos 3\varphi_1) \cos \varphi_1 =$$
$$= 3 - (\cos 3\varphi_1 \cos \varphi_1 + \sin 3\varphi_1 \sin \varphi_1) = 3 - \cos (3\varphi_1 - \varphi_1) =$$
$$= 3 - \cos 2\varphi_1 .$$

Thus, the parametric equations of the line of action go as follows:

$$X_k = -\sin 2\varphi_1 , \quad Y_k = 3 - \cos 2\varphi_1 . \tag{48}$$

The parameter φ_1 can be cancelled in the following way:

$$X_k = -\sin 2\varphi_1 , \quad Y_k - 3 = -\cos 2\varphi_1 .$$

By taking the squares and adding them, we obtain:

$$X_k^2 + (Y_k - 3)^2 = 1.$$

Consequently, the line of action represents an arc as a portion of a circle as shown in Fig. 14. This is quite underst-andable, because of the following relationship: when profile (p_2) moves around centre O_2, we find a right-angled triangle formed by the hypotenuse O_2C and the sides PC and O_2P where PC represents a portion of the normal to the profile coordinated to point P, and O_2P is the radius. Consequently, point P representing the right-angle corner moves along a circular arc.

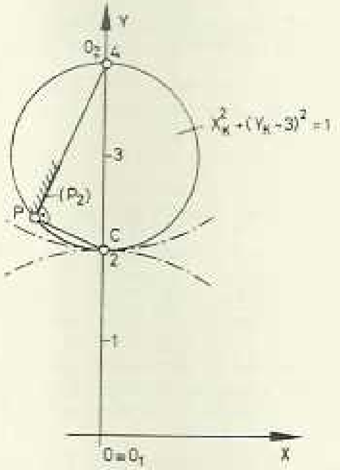

Fig. 14

The gear ratio variation which occurs when the original centre distance a gets altered should be examined; e.g. we allow the value a_0 to be decreased to $a = a_0 - \Delta a$. In Fig. 15 the dash curves represent the initial position of the mating profiles when distance a is kept unchanged and ratio i remains constant. (In this position, C represents the mating contact point.)

The centre distance a should be modified inasmuch as centre O_1 remains in the original position, whereas centre O_2 comes nearer to O_1 by the decrement Δa, and reaches the new position O_2^*.

When the two gears have performed some revolution through the angles λ_1 and λ_2 respectively (measured from the initial position), the new mating point is at P^*. In this position, the normal to the profile at the new pitch point will not produce the same division ratio of the centre distance a as was with the original distance a_0; thus the gear ratio i is also subjected to some change, characterized by the new dividing point C^*. Now, we return to Eqs (23) by which the mating conditions are established, but we substitute λ_1 instead of t, and λ_2 instead of T.

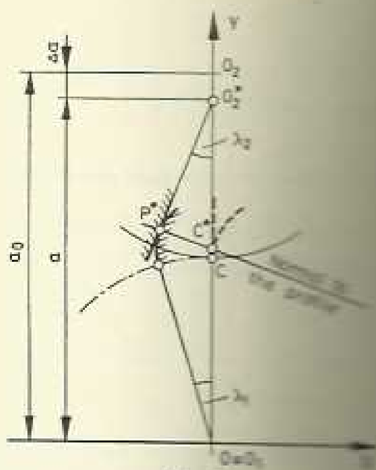

Fig. 15

For this purpose, we apply:

$$X_{01} = 3 \sin \varphi_1 - \sin 3\varphi_1 , \qquad X_{02} = 0,$$
$$Y_{01} = 3 \cos \varphi_1 - \cos 3\varphi_1 , \qquad Y_{02} = -2 \cos \varphi_1 ,$$

and

$$X'_{01} = 3(\cos \varphi_1 - \cos 3\varphi_1), \qquad X'_{02} = 0,$$
$$Y'_{01} = 3(-\sin \varphi_1 + \sin 3\varphi_1), \qquad Y'_{02} = 2 \sin \varphi_1 .$$

Thus, Eqs (23) are rewritten:

$$(3 \sin \varphi_1 - \sin 3\varphi_1) \cos \lambda_1 - (3 \cos \varphi_1 - \cos 3\varphi_1) \sin \lambda_1 = -2 \cos \varphi_1 \sin \lambda_2 ,$$

$$(3 \sin \varphi_1 - \sin 3\varphi_1) \sin \lambda_1 + (3 \cos \varphi_1 - \cos 3\varphi_1) \cos \lambda_1 = a - 2 \cos \varphi_1 \cos \lambda_2 , \tag{7/23}$$

$$\tan \lambda_2 = \frac{3(\cos \varphi_1 - \cos 3\varphi_1)\, 2 \sin \varphi_1 - 3(-\sin \varphi_1 + \sin 3\varphi_1)\, 2 \sin \varphi_1 \tan \lambda_1}{3(\cos \varphi_1 - \cos 3\varphi_1)\, 2 \sin \varphi_1 \tan \lambda_1 + 3(-\sin \varphi_1 + \sin 3\varphi_1)\, 2 \sin \varphi_1} .$$

Some kind of simplification seems to be possible on account of the parameter φ_1 being the same for both profiles. The 3rd equation among Eqs (7/23) can be rearranged. There is a simplification by 6 and by $\sin \varphi_1$ possible, and the known relations

$$\cos \varphi_1 - \cos 3\varphi_1 = 4 \cos \varphi_1 \sin^2 \varphi_1 = 2 \sin \varphi_1 \sin 2\varphi_1 ,$$
$$-\sin \varphi_1 + \sin 3\varphi_1 = 2 \sin \varphi_1 \cos 2\varphi_1$$

can be applied:

$$\tan \lambda_2 = \frac{2 \sin \varphi_1 \sin 2\varphi_1 - 2 \sin \varphi_1 \cos 2\varphi_1 \tan \lambda_1}{2 \sin \varphi_1 \sin 2\varphi_1 \tan \lambda_1 + 2 \sin \varphi_1 \cos 2\varphi_1} .$$

Both numerator and denominator can be divided by

$$2 \sin \varphi_1 \cos 2\varphi_1$$

to obtain:

$$\tan \lambda_2 = \frac{\tan 2\varphi_1 - \tan \lambda_1}{\tan 2\varphi_1 \tan \lambda_1 + 1} = \tan(2\varphi_1 - \lambda_1)$$

$$\lambda_2 = 2\varphi_1 - \lambda_1 , \qquad \lambda_1 + \lambda_2 = 2\varphi_1 . \tag{49}$$

We are able to transform the 1st and the 2nd formulae among Eqs (7/23):

$$3 \sin \varphi_1 \cos \lambda_1 - \sin 3\varphi_1 \cos \lambda_1 - 3 \cos \varphi_1 \sin \lambda_1 + \cos 3\varphi_1 \sin \lambda_1 = -2 \cos \varphi_1 \sin \lambda_2 ,$$
$$3 \sin \varphi_1 \sin \lambda_1 - \sin 3\varphi_1 \sin \lambda_1 + 3 \cos \varphi_1 \cos \lambda_1 - \cos 3\varphi_1 \cos \lambda_1 = a - 2 \cos \varphi_1 \cos \lambda_2 ,$$
$$3 \sin (\varphi_1 - \lambda_1) - \sin (3\varphi_1 - \lambda_1) = -2 \cos \varphi_1 \sin \lambda_2 ,$$
$$3 \cos (\varphi_1 - \lambda_1) - \cos (3\varphi_1 - \lambda_1) = a - 2 \cos \varphi_1 \cos \lambda_2 .$$

We form the square of both sides and add them together

$$10 - 6 \cos 2\varphi_1 = 4 \cos^2 \varphi_1 - 4a \cos \varphi_1 \cos \lambda_2 + a^2,$$

or

$$\cos \lambda_2 = \frac{2 \cos^2 \varphi_1 + 6 \cos 2\varphi_1 + a^2 - 10}{4a \cos \varphi_1} . \tag{50}$$

In a numerical example, we put

$$\Delta a = -0.1, \quad \text{viz. } a = 3.9,$$

thus,

$$\cos \lambda_2 = \frac{4 \cos^2 \varphi_1 - 6 \cos 2\varphi_1 + 5.21}{15.6 \cos \varphi_1} . \tag{51}$$

With various values of φ_1 and by using Eq. (49) the angles λ_1 or, by Eq. (51), the angles λ_2 are calculated. The resulting values (φ_1, λ_1, λ_2) give us every required point of contact, at the same time being the mating point in a position when gear (2) revolved through angle λ_2. The actual normal to the profile passes through the resulting point P^* (X^*, Y^*), and its equation goes as follows:

$$Y - Y^* = m (X - X^*),$$

where

$$X^* = -2 \cos \varphi_1 \sin \lambda_2 ,$$
$$Y^* = 3.9 - 2 \cos \varphi_1 \cos \lambda_2 ,$$
$$m = -\tan \lambda_2 .$$

Now we find the co-ordinates of point C^* which represents the intersection point of the normal to the profile with the centre line O_1O_2 : ($X^* = 0$).

$$Y_{C*} = Y^* + \tan \lambda_2 X^* = O_1C^*, \tag{52}$$

and

$$i = \frac{\overline{O_2C^*}}{O_1C^*} \tag{53}$$

represents the momentaneous value of the gear ratio. Our numerical results are shown in the following Table:

φ_1	λ_2	λ_1	X^*	Y^*	O_1C^*	O_2C^*	i
0°	−12°50′10″	12°50′10″	−0,444	1,950	1,849	2,051	1,109
1°	−12°52′50″	14°52′50″	−0,446	1,951	1,850	2,050	1,108

and so forth.

For calculating the relative sliding velocity we shall return to Eq. (29).

Let the given centre distance a equal 4 and the angular velocity ω_1 equal 1; thus, the expression as exposed under the square root sign in (29), of course, with $t = \varphi_1$ [according to (47)] has to be written as:

$$|v_r|^2 = 4^2 - 2 \cdot 4\,(1+1)\,[(3 \sin\varphi_1 - \sin 3\varphi_1)\sin\varphi_1 + (3\cos\varphi_1 - \cos 3\varphi_1)\cos\varphi_1 + $$
$$+ (1+1)^2\,[(3\sin\varphi_1 - \sin 3\varphi_1)^2 + (3\cos\varphi_1 - \cos 3\varphi_1)^2] = \frac{16\,(1-\cos 2\varphi_1)}{2},$$

or

$$|v_r| = \sqrt{\frac{16\,(1-\cos 2\varphi_1)}{2}} = 4\sqrt{\frac{1-\cos 2\varphi_1}{2}} = 4\sin\varphi_1 . \tag{7/29}$$

Calculation of the normal tooth force according to to Eq. (46):

$$F_n = M_1 \frac{\sqrt{[3\,(\cos\varphi_1 - \cos 3\varphi_1)]^2 + [3\,(-\sin\varphi_1 + \sin 3\varphi_1)]^2}}{(3\sin\varphi_1 - \sin 3\varphi_1)\,3\,(\cos 3\varphi_1 - \cos 3\varphi_1) + (3\cos\varphi_1 - \cos 3\varphi_1)\,3(-\sin\varphi_1 + \sin 3\varphi_1)}$$

where the expression under the square root sign is reduced to:

$$36 \sin^2\varphi_1 ,$$

thus, the numerator equals $= 6 \sin\varphi_1$ and the denominator is reduced to

$$12 \sin\varphi_1 \cos\varphi_1 ,$$

and consequently:

$$F_n = M_1 \frac{6\sin\varphi_1}{12\sin\varphi_1\cos\varphi_1} = M_1 \frac{1}{2\cos\varphi_1} . \tag{7/40}$$

7.2 *Case of a special application*

7.2.1. *Example of a cam profile*

This case is shown in Fig. 16. Here, the profile of member (1) is characterized as a straight line passing through centre O_1, and the profile of member (2) represents a circle having the radius r which swings, with a connecting lever of the length m, around centre O_2; the possibility that this circle can, at the same time rotate, around centre M has no mechanical significance. The mechanism as shown in Fig. 16, assumes its initial position. For profile (1), the equation in the system of co-ordinates $x_1 y_1$ can be written as:

$$\begin{aligned} X_{01} &= \varphi_1 \cos\alpha \qquad & X'_{01} &= \cos\alpha \\ Y_{01} &= \varphi_1 \sin\alpha \qquad & Y'_{01} &= \sin\alpha \end{aligned} \tag{54}$$

Further, φ_1 represents the distance of the moving point of profile (1) from centre O_1 and the position of this straight profile is characterized by

$$\cos\alpha = \frac{r}{a-m}, \tag{55}$$

$$\sin\alpha = \sqrt{1 - \frac{r^2}{(a-m)^2}} = \frac{\sqrt{(a-m)^2 - r^2}}{a-m} .$$

As for the profile (2), expressed in the system of co-ordinates $x_2 y_2$, we can write:

$$\begin{aligned} X_{02} &= r\cos\varphi_2 , \qquad & X'_{02} &= -r\sin\varphi_2 , \\ Y_{02} &= (m + r\sin\varphi_2), \qquad & Y'_{02} &= -r\cos\varphi_2 \end{aligned} \tag{56}$$

where angle φ_2 is to be understood in accordance with Fig. 16.

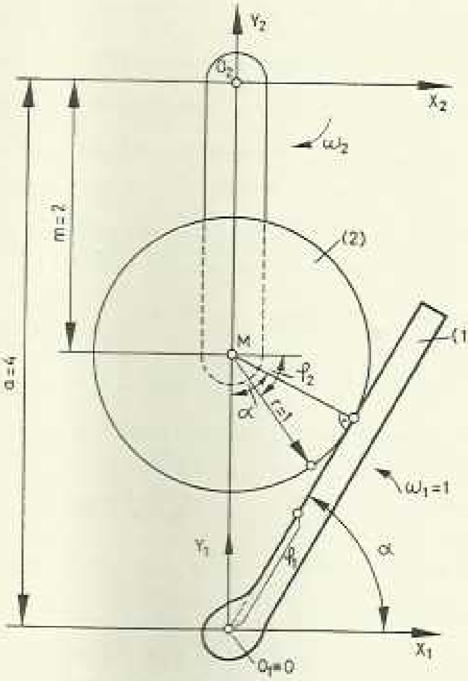

Fig. 16

Suppose that member (1) rotates around centre O_1 in the positive sense at the angular velocity $\omega_1 = 1$.

What is required, is to know the gear ratio $\omega_1/\omega_2 = i$:

Recurring to Eqs (23) we can write

$$\varphi_1 \cos \alpha \cos t - \varphi_1 \sin \alpha \sin t = r \cos \varphi_2 \cos T - (m + r \sin \varphi_2) \sin T$$

$$\varphi_1 \cos \alpha \sin t + \varphi_1 \sin \alpha \cos t = a - r \cos \varphi_2 \sin T - (m + r \sin \varphi_2) \cos T$$

$$\tan T = \frac{(-\cos \alpha \, r \cos \varphi_2 - r \sin \varphi_2 \sin \alpha) - (-r \cos \alpha \sin \varphi_2 - r \sin \alpha \cos \varphi_2) \tan t}{(-\cos \alpha \, r \cos \varphi_2 + r \sin \varphi_2 \sin \alpha) \tan t + (-r \cos \alpha \sin \varphi_2 - r \sin \alpha \cos \varphi_2)} \,. \quad (57)$$

After simplification and rearrangement:

$$\begin{aligned} \varphi_1 \cos (\alpha + t) &= r \cos (\varphi_2 + T) - m \sin T, \\ \varphi_1 \sin (\alpha + t) &= a - r \sin (\varphi_2 + T) - m \cos T, \\ \tan T &= \frac{\sin (\varphi_2 + \alpha) \tan t - \cos (\varphi_2 + \alpha)}{-\sin (\varphi_2 + \alpha) - \cos (\varphi_2 + \alpha) \tan t} \,. \end{aligned} \quad (58)$$

In the latter equation an amplification by -1 seems to be suitable:

$$\tan T = \frac{\cos (\varphi_2 + \alpha) - \sin (\varphi_2 + \alpha) \tan t}{\sin (\varphi_2 + \alpha) + \cos (\varphi_2 + \alpha) \tan t} \,,$$

and after a second amplification by $1/\cos (\varphi_2 + \alpha)$, we obtain

$$\tan T = \frac{1 - \tan (\varphi_2 + \alpha) \tan t}{\tan (\varphi_2 + \alpha) + \tan t} = \frac{1}{\tan [(\varphi_2 + \alpha) + t]} =$$

$$= \cot [(\varphi_2 + \alpha) + t] = \tan \{90^\circ - [(\varphi_2 + \alpha) + t]\}.$$

or

$$T = 90^\circ - [(\varphi_2 + \alpha) + t] = 90^\circ - \varphi_2 - \alpha - t$$

$$\varphi_2 + T = 90^\circ - \alpha - t. \tag{59}$$

By substituting this formula into the 1st and 2nd Eqs (58) we obtain:

$$\varphi_1 \cos(\alpha + t) = r \cos[90^\circ - (\alpha + t)] - m \sin T,$$

$$\varphi_1 \sin(\alpha + t) = a - r \sin[90^\circ - (\alpha + t)] - m \cos T,$$

or

$$\varphi_1 \cos(\alpha + t) = r \sin(\alpha + t) - m \sin T, \tag{60}$$

$$\varphi_1 \sin(\alpha + t) = a - r \cos(\alpha + t) - m \cos T.$$

Dividing the latter equation by the former we have:

$$\tan(\alpha + t) = \frac{a - r\cos(\alpha + t) - m\cos T}{r\sin(\alpha + t) - m\sin T}$$

that sets down a relation between t and T. After cancellation of the fractions and rearrangement, we can write:

$$\frac{r - m\sin(\alpha + t)\sin T}{\cos(\alpha + t)} - a = -m\sqrt{1 - \sin^2 T}$$

$$r - m\sin(\alpha + t)\sin T - a\cos(\alpha + t) = -m\cos(\alpha + t)\sqrt{1 - \sin^2 T}\,.$$

Forming the squares of both sides:

$$r^2 + m^2\sin^2(\alpha + t)\sin^2 T + a^2\cos^2(\alpha + t) - 2rm\sin(\alpha + t)\sin T -$$

$$- 2ar\cos(\alpha + t) + 2am\sin(\alpha + t)\cos(\alpha + t)\sin T =$$

$$= m^2\cos^2(\alpha + t) - m^2\cos^2(\alpha + t)\sin^2 T$$

and rearranged:

$$m^2\sin^2 T + 2m\sin(\alpha + t)[a\cos(\alpha + t) - r]\sin T + \{[r - a\cos(\alpha + t)]^2 - m^2\cos^2(\alpha + t)\} = 0.$$

By using the root-formula of a second-degree equation, $\sin T$ can be calculated.

First, the discriminant should be expanded:

$$4m^2\sin^2(\alpha + t)[a\cos(\alpha + t) - r]^2 - 4m^2\{[a\cos(\alpha + t) - r]^2 - m^2\cos^2(\alpha + t)\} =$$

$$= 4m^2\cos^2(\alpha + t)\{m^2 - [a\cos(\alpha + t) - r]^2\},$$

and consequently

$$\sin T = \frac{[r - a\cos(\alpha + t)]\sin(\alpha + t) + \cos(\alpha + t)\sqrt{m^2 - [a\cos(\alpha + t) - r]^2}}{m} \tag{61}$$

of course, we apply the positive root.

With the numeric data:

$$a = 4, \quad r = 1, \quad m = 2$$

we have:

$$\alpha = \arccos\frac{r}{a - m} = \arccos\frac{1}{4 - 2} = \arccos\frac{1}{2},$$

$$\alpha = \frac{\pi}{3}\,,$$

and

$$\sin T = \frac{[1 - 4\cos(t + \pi/3)]\sin(t + \pi/3) + \cos(t + \pi/3)\sqrt{4 - [4\cos(t + \pi/3) - 1]^2}}{2}\,. \tag{62}$$

From this we obtain the angular revolution of the gear member (2) through the time t:

$$T = \arc\sin \frac{[1 - 4\cos(t+\pi/3)]\sin(t+\pi/3) + \cos(t+\pi/3)\sqrt{4 - [4\cos(t+\pi/3) - 1]^2}}{2}$$

and with

$\omega_2 = T' =$

$$= \frac{1}{\sqrt{1 - \left\{\frac{[1 - 4\cos(t+\pi/3)]\sin(t+\pi/3) + \cos(t+\pi/3)\sqrt{4 - [\cos(t+\pi/3) - 1]^2}}{2}\right\}^2}} \cdot$$

$$\cdot \frac{1}{2}\Bigg\{4\sin^2(t+\pi/3) + [1 - 4\cos(t+\pi/3)]\cos(t+\pi/3) -$$

$$- \sin(t+\pi/3)\sqrt{4 - [4\cos(t+\pi/3) - 1]^2} +$$

$$+ \cos(t+\pi/3)\frac{2\,[4\cos(t+\pi/3) - 1]\,4\sin(t+\pi/3)}{2\sqrt{4 - [4\cos(t+\pi/3) - 1]^2}}\Bigg\}.$$

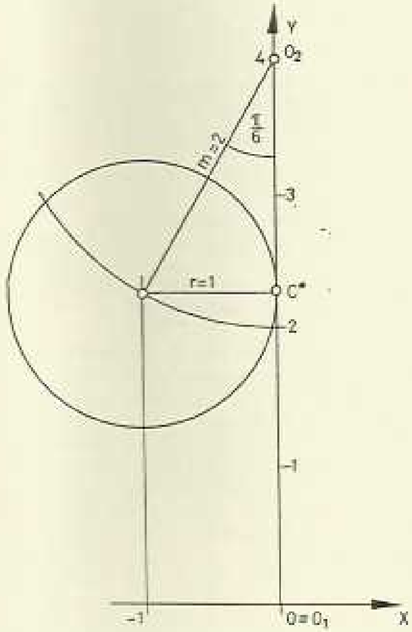

Fig. 17

Instead of a general reduction of this equation, we may confine ourselves to the determination of the gear ratio at a certain point of time, say at $t = \pi/6$.

Thus,

$$\omega_2(\pi/6) = \frac{4 - \sqrt{3}}{\sqrt{3}},$$

and

$$i = \frac{1}{\omega_2} = \frac{\sqrt{3}}{4 - \sqrt{3}} \approx 0{,}76.$$

The same result can be obtained by using Eqs (62). First we can state that after a revolution through angle $t = \pi/6$ in which position the edge of member (1) comes into coincidence with axis y, we obtain

$$\sin T = 1/2,$$

$$T = \arcsin 1/2 = \pi/6,$$

in words, the member (2) likewise performed a revolution through an angle $T = \pi/6$. In this position, the profile normal passes through point C^*. As can be seen from Fig. 17, this portion is characterized by the relation:

$$\overline{O_2C^*} = 2 \cos \pi/6 = \frac{2\sqrt{3}}{2} = \sqrt{3},$$

$$\overline{O_1C^*} = 4 - \sqrt{3},$$

and the momentaneous value of the gear ratio:

$$i = \frac{\overline{O_2C^*}}{\overline{O_1C^*}} = \frac{\sqrt{3}}{4-\sqrt{3}} \sim 0.76$$

in conformity with our first result.

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Komplexe Schreibweise im Entwerfen von Zahnrädern. Hier wollen wir mit besonderer Rücksicht auf Grundfragen des Entwerfens einige geometrische und mechanische Eigenschaften von geradverzahnten zylindrischen Räderpaaren besprechen, u. zw.: Bestimmung des Gegenprofils zu einem gegebenen; Untersuchung der Eingriffslinie; Analyse der geometrischen und mechanischen Bedingungen des Eingriffs; Untersuchung der infolge eines Achsenabstandfehlers entstandenen Übersetzungsänderung; Berechnung der relativen Gleitgeschwindigkeit der Zähne; Berechnung der momentanen, normalen Zahnkraft. Bei gegebenem Profil eines der beiden Zahnräder (und dies ist gewöhnlich das kleine Rad) erscheint die Eingriffslinie eindeutig bestimmt, und dasselbe gilt für das Profil des großen Rades. Die hier bekanntgemachten Rechnungsmethoden sind — nebst der komplexen Ausdrucksweise — dadurch gekennzeichnet, daß die Ergebnisse allgemein mit Hilfe von den, das Profil des kleineren Rades bestimmenden Funktionen ausgedrückt werden.

Комплексное представление зубчатых колес при их проектировании. Данная работа занимается такими геометрическими и механическими свойствами пар прямозубых зубчатых колес с параллельными осями, которые при проектировании зубьев играют основную роль, а именно: определение контрпрофиля, соответствующего данному профилю; анализ контактной линии; анализ геометрических и механических условий контакта (зацепления); анализ изменения соотношения передачи, возникающего вследствие погрешности расстояния между осями; вычисление относительной скорости скольжения зубьев и мгновенного значения усилия зубьев в нормальном направлении. Принимая заданной профиль одного колеса (обычно меньшего), это однозначно определяет форму контактной линии, а эта форма же определяет форму контрпрофиля; одним из характерных черт описываемых в дальнейшем методик расчета наряду с комплексным представлением является то, что результаты вообще выражают с помощью данных функций, описывающих профиль малого колеса.