TORSIONAL DEFORMATION OF COMPOUND CIRCULAR BARS

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Abstract

Compound circular bar consists of two bars with same cross sections and different homogeneous isotropic elastic materials. The bond between the bar components is perfect. The free end cross sections of the bar components are subjected to torsional shear stresses. Closed-form solutions are presented for circumferential displacement, longitudinal and radial shearing stresses.

1. INTRODUCTION

The considered compound circular bar is shown in Fig. 1. The space domain B is occupied by the compound bar in cylindrical coordinate system $Or\varphi z$ can be discribed as (Fig. 1) $B = B_1 \cup B_2$, $B_i = \{(r,\varphi,z)\,|\,0 \le r \le R,\ 0 \le \varphi \le 2\pi,\ l_{i1} \le z \le l_{i2}\}$ $(i = 1,2)$, where $l_{11} = -L_1$, $l_{12} = l_{21} = 0$, $l_{22} = L_2$. The cylindrical boundary surface $\partial B_3 = \{(r,\varphi,z)\,|\,r = R,\ 0 \le \varphi \le 2\pi,\ -L_1 \le z \le L_2\}$ is traction free. The end cross section $\partial B_i = \{(r,\varphi,z)\,|\,0 \le r \le R,\ 0 \le \varphi \le 2\pi,\ z = l_i\}$ $(i = 1,2)$, where $l_1 = -L_1$, $l_2 = L_2$, are loaded by axially symmetric torsional loads that is

$$\tau_{r\varphi}(r,\varphi,-L_1) = f_1(r), \quad \tau_{r\varphi}(r,\varphi,L_2) = f_2(r), \quad 0 \le r \le R, \quad 0 \le \varphi \le 2\pi. \tag{1}$$

The corresponding torque to the applied torsional loads given by Eq. (1) is as follows (Fig. 1)

$$T = 2\pi \int_0^R r^2 f_1(r)\mathrm{d}r = 2\pi \int_0^R r^2 f_2(r)\mathrm{d}r. \tag{2}$$

The torsional problem defined above can be solved by the use of Michell-Föppl's theory of torsion of body of rotation [1-5]. In the Michell-Föppl's theory of torsion it is assumed that the displacement field of body of rotation has the next form

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z, \quad u = w = 0, \quad v = v(r,z). \tag{3}$$

where $\mathbf{e}_r$, $\mathbf{e}_\varphi$ and $\mathbf{e}_z$ are the unit vectors of the cylindrical coordinate system $Or\varphi z$ (Fig.1.). Following Lekhnitskii [4] we introduce the next auxiliary function $\psi(r,z) = v(r,z)/r$. The strains and stresses in terms of $\psi = \psi(r,z)$ can be expressed as

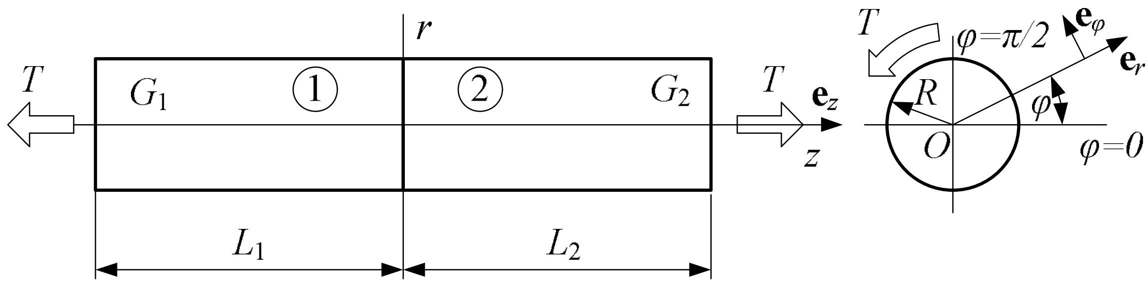

Figure 1. Compound circular bar with torsional load.

$$\gamma_{r\varphi} = r\frac{\partial\psi}{\partial r}, \quad \gamma_{z\varphi} = r\frac{\partial\psi}{\partial z}, \quad \tau_{r\varphi} = G(z)r\frac{\partial\psi}{\partial r}, \quad \tau_{z\varphi} = G(z)r\frac{\partial\psi}{\partial z}. \tag{4}$$

In Eq. (4) $\gamma_{r\varphi}$, $\gamma_{z\varphi}$ are the shear strains, $\tau_{r\varphi}$, $\tau_{z\varphi}$ are the shearing stresses $G = G(z)$ is the shear modulus which can be represented in terms of shear moduli of bar components 1 and 2 in the next form (Fig. 1)

$$G(z) = \left(H(z+L_1) - H(z)\right)G_1 + H(z)G_2, \quad 0 \le r \le R, \quad -L_1 \le z \le L_2. \tag{5}$$

In Eq. (5) $H = H(z)$ is the Heaviside unit step function defined as

$$H(z-a) = \begin{cases} 1 \ \text{for} \ z > a, \\ 0 \ \text{for} \ z < a. \end{cases} \tag{6}$$

The equilibrium equation for shear stresses in case of torsional deformation of body of rotation is as follows

$$\frac{\partial}{\partial r}\left(r^2\tau_{r\varphi}\right) + \frac{\partial}{\partial z}\left(r^2\tau_{z\varphi}\right) = 0, \quad 0 \le r \le R, \quad -L_1 \le z \le L_2. \tag{7}$$

Substitution of Eqs. $(4)_{3,4}$ into Eq. (7) gives

$$\frac{\partial^2\psi_i}{\partial r^2} + \frac{3}{r}\frac{\partial\psi_i}{\partial r} + \frac{\partial^2\psi_i}{\partial z^2} = 0, \quad 0 \le r \le R, \quad l_{i1} \le z \le l_{i2}, \quad (i = 1,2). \tag{8}$$

The boundary conditions for the considered torsion problem can be written in the next form

$$\frac{\partial\psi_i}{\partial r} = 0, \quad r = R, \quad l_{i1} \le z \le l_{i2}, \quad (i = 1,2), \tag{9}$$

$$G_1 r\frac{\partial\psi_1}{\partial z} = f_1(r), \quad z = -L_1, \quad 0 \le r \le R, \tag{10}$$

$$G_2 r\frac{\partial\psi_2}{\partial z} = f_2(r), \quad z = L_2, \quad 0 \le r \le R. \tag{11}$$

At the connection of bar components 1 and 2 the join condition must be satisfied

$$\psi_1(r,0)=\psi_2(r,0),\quad G_1\left.\frac{\partial\psi_1}{\partial z}\right|_{z=0}=G_2\left.\frac{\partial\psi_2}{\partial z}\right|_{z=0},\quad 0\le r\le R. \tag{12}$$

Equation $(12)_1$ expressed the continuity condition of diplacement field and Eq. $(12)_2$ formulates the continuity condition of shearing stress $\tau_{z\varphi}$ at $z=0$.

2. SOLUTION OF TORSION PROBLEM

At first, we introduce the dimensionless radial and axial coordinates by the next definition

$$\rho=\frac{r}{R},\quad \zeta=\frac{z}{R},\quad 0\le\rho\le1,\quad \zeta_1=-\frac{L_1}{R}\le\zeta\le\zeta_2=\frac{L_2}{R}. \tag{13}$$

Equations (8), $(4)_{3,4}$ and Eqs. (9–12) can be reformulated as

$$\frac{\partial^2\psi_i}{\partial\rho^2}+\frac{3}{\rho}\frac{\partial\psi_i}{\partial\rho}+\frac{\partial^2\psi_i}{\partial\zeta^2}=0,\quad 0\le\rho\le1,\quad \zeta_{1i}\le\zeta\le\zeta_{2i},\quad (i=1,2), \tag{14}$$

$$\zeta_{11}=\zeta_1,\quad \zeta_{21}=\zeta_{12}=0,\quad \zeta_{22}=\zeta_2, \tag{15}$$

$$\tau_{r\varphi}=G_i\rho\frac{\partial\psi_i}{\partial\rho},\quad \tau_{z\varphi}=G_i\rho\frac{\partial\psi_i}{\partial\zeta},\quad 0\le\rho\le1,\quad \zeta_{1i}\le\zeta\le\zeta_{2i},\ (i=1,2), \tag{16}$$

$$\frac{\partial\psi_i}{\partial\rho}=0,\quad \rho=1,\quad \zeta_{1i}\le\zeta\le\zeta_{2i},\quad (i=1,2), \tag{17}$$

$$G_1\rho\frac{\partial\psi_1}{\partial\rho}=f_1(\rho),\quad \zeta=\zeta_1,\quad 0\le\rho\le1, \tag{18}$$

$$G_2\rho\frac{\partial\psi_2}{\partial\rho}=f_2(\rho),\quad \zeta=\zeta_2,\quad 0\le\rho\le1, \tag{19}$$

$$\psi_1(\rho,0)=\psi_2(\rho,0),\quad G_1\left.\frac{\partial\psi_1}{\partial\zeta}\right|_{\zeta=0}=G_2\left.\frac{\partial\psi_2}{\partial\zeta}\right|_{\zeta=0},\quad 0\le\rho\le1. \tag{20}$$

The numerical solution will be presented for the next type of torsional loads

$$f_1(\rho)=f_2(\rho)=K\left(6\rho^3-5\rho^2\right)+\frac{2T}{R^3\pi}\rho. \tag{21}$$

In Eq. (21) K is a constant, and it can be shown that torsional shear load

$$\hat{f}_1(r)=\hat{f}_2(r)=K\left[6\left(\frac{r}{R}\right)^3-5\left(\frac{r}{R}\right)^2\right] \tag{22}$$

does not produce torsional moment, that is we have

$$2\pi\int_0^R r^2 \hat{f}_1(r)\mathrm{d}r = 0. \tag{23}$$

The solution of the boundary value problem formulated in Eqs. (14–20) for $\psi_i = \psi_i(\rho,\zeta)$ $(i=1,2)$ can be written in the form

$$\psi_1 = \psi_1(\rho,\zeta) = c_1\zeta + \sum_{k=1}^{\infty}\frac{J_1(\lambda_k\rho)}{\rho}\left(a_k\cosh\lambda_k\zeta + b_k\sinh\lambda_k\zeta\right), \quad 0\le\rho\le 1, \quad \zeta_1\le\zeta\le 0, \tag{24}$$

$$\psi_2 = \psi_2(\rho,\zeta) = c_2\zeta + \sum_{k=1}^{\infty}\frac{J_1(\lambda_k\rho)}{\rho}\left(a_k\cosh\lambda_k\zeta + \frac{G_1}{G_2}b_k\sinh\lambda_k\zeta\right), \\ 0\le\rho\le 1, \quad 0\le\zeta\le\zeta_2. \tag{25}$$

In Eqs. (24–25)

$$c_1 = \frac{2T}{G_1R^3\pi}, \quad c_2 = \frac{2T}{G_2R^3\pi}, \tag{26}$$

$$a_k = \frac{q_k}{\lambda_kG_2}\frac{\cosh\lambda_k\zeta_2 - \cosh\lambda_k\zeta_1}{\frac{G_1}{G_2}\cosh\lambda_k\zeta_2\sinh\lambda_k\zeta_1 - \cosh\lambda_k\zeta_1\sinh\lambda_k\zeta_2}, \tag{27}$$

$$b_k = \frac{q_k}{\lambda_k}\frac{\frac{\sinh\lambda_k\zeta_1}{G_2} - \frac{\sinh\lambda_k\zeta_2}{G_1}}{\frac{G_1}{G_2}\cosh\lambda_k\zeta_2\sinh\lambda_k\zeta_1 - \cosh\lambda_k\zeta_1\sinh\lambda_k\zeta_2}, \tag{28}$$

$$q_k = \frac{2\lambda_k^2\int_0^1 \rho\hat{f}(\rho)J_1(\lambda_k\rho)\mathrm{d}\rho}{\lambda_k^2\left[J_1'(\lambda_k)\right]^2 + \left(\lambda_k^2-1\right)\left[J_1(\lambda_k)\right]^2}, \tag{29}$$

here $J_1(x)$ is the Bessel function of the first kind, of first order and

$$J_1'(x) = \frac{\mathrm{d}J_1(x)}{\mathrm{d}x}, \tag{30}$$

furthermore λ_k $(k=1,2,\ldots;\ \lambda_1<\lambda_2<\lambda_3<\ldots)$ is the root of the next transcendent equation

$$xJ_0(x) - 2J_1(x) = 0. \tag{31}$$

The radial shearing stress is obtained from Eq. $(16)_1$

$$\tau_{r\varphi}=\tau_{r\varphi}(\rho,\zeta)=G_1\sum_{k=1}^{\infty}\left(\lambda_k J_1'(\lambda_k\rho)-\frac{J_1(\lambda_k\rho)}{\rho}\right)\left(a_k\cosh\lambda_k\zeta+b_k\sinh\lambda_k\zeta\right),\qquad(32)$$
$$0\le\rho\le 1,\quad \zeta_1\le\zeta\le 0,$$

$$\tau_{r\varphi}=\tau_{r\varphi}(\rho,\zeta)=G_2\sum_{k=1}^{\infty}\left(\lambda_k J_1'(\lambda_k\rho)-\frac{J_1(\lambda_k\rho)}{\rho}\right)\left(a_k\cosh\lambda_k\zeta+\frac{G_1}{G_2}b_k\sinh\lambda_k\zeta\right),\qquad(33)$$
$$0\le\rho\le 1,\quad 0\le\zeta\le\zeta_2.$$

The formula of the longitudinal shearing stresses can be derived by the use of Eq. $(16)_2$

$$\tau_{z\varphi}=\tau_{z\varphi}(\rho,\zeta)=G_1\left(c_1\rho+\sum_{k=1}^{\infty}\lambda_k J_1(\lambda_k\rho)\left(a_k\sinh\lambda_k\zeta+b_k\cosh\lambda_k\zeta\right)\right),\qquad(34)$$
$$0\le\rho\le 1,\quad \zeta_1\le\zeta\le 0,$$
$$\tau_{z\varphi}=\tau_{z\varphi}(\rho,\zeta)=G_2\left(c_2\rho+\sum_{k=1}^{\infty}\lambda_k J_1(\lambda_k\rho)\left(a_k\sinh\lambda_k\zeta+\frac{G_1}{G_2}b_k\cosh\lambda_k\zeta\right)\right),\qquad(35)$$
$$0\le\rho\le 1,\quad 0\le\zeta\le\zeta_2.$$

3. NUMERICAL EXAMPLE

In the numerical example the next data are used: $R=5$ mm, $L_1=L_2=20$ mm, $G_1=95\times10^3$ MPa, $G_2=35\times10^3$ MPa, $T=10^4$ Nmm, $K=-8\times10^2$ MPa. In Figure 2 the graph of $\psi=\psi(1,\zeta)$ is shown. Figure 3 illustrates the graphs of $\tau_{z\varphi}=\tau_{z\varphi}(\rho,\zeta)$ for $\zeta=\zeta_1$, $\zeta=\zeta_1+0.05$, $\zeta=\zeta_1+0.25$, $\zeta=\zeta_1+0.5$, $\zeta=\zeta_1+1$, $\zeta=\zeta_1+1.5$, $0\le\rho\le1$. Figure 3 also illustrates that the effect of torsional load $\hat{f}=\hat{f}(r)$ is significant only near the left end cross section. The plots of $\tau_{r\varphi}=\tau_{r\varphi}(\rho,\zeta)$ for $\zeta=\zeta_1$, $\zeta=\frac{\zeta_1}{3}$, $\zeta=0$, $\zeta=\frac{\zeta_2}{3}$, $\zeta=\zeta_2$ as a function of ρ $(0\le\rho\le1)$ are shown in Fig. 4. The torsional moment as the fucntion of ζ obtained from next equation

$$t(\zeta)=2\pi\int_0^R r^2\tau_{r\varphi}(r,z)\mathrm{d}r=2\pi R^3\int_0^1\rho^2\tau_{z\varphi}(\rho,\zeta)\mathrm{d}\rho\qquad(36)$$

is illustrated in Fig. 5. In formula (36) $\tau_{z\varphi}=\tau_{z\varphi}(\rho,\zeta)$ is given by Eqs. (34–35). According to the equilibrium condition of twisted axially compound bar we have $t(\zeta)=\text{constant}=10^4$ Nmm for $\zeta_1\le\zeta\le\zeta_2$. The graph of $t(\zeta)$ supported the validity of the presented analytical solution.

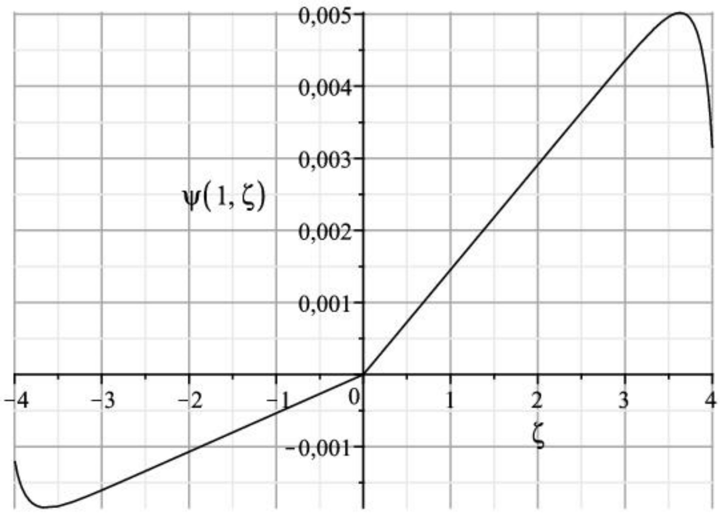

Figure 2. The graph of $\psi(1,\zeta)$.

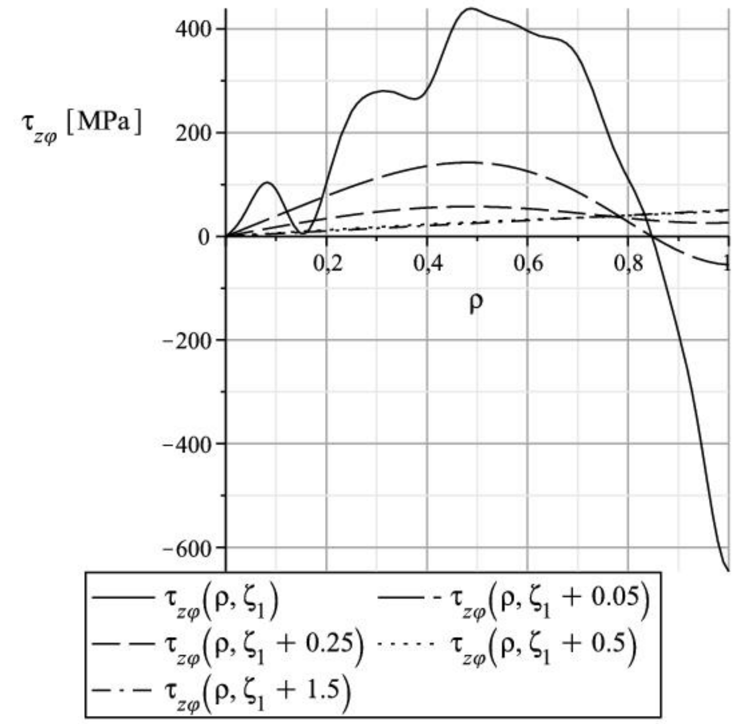

Figure 3. The graph of $\tau_{z\varphi}(\rho,\zeta_i)$.

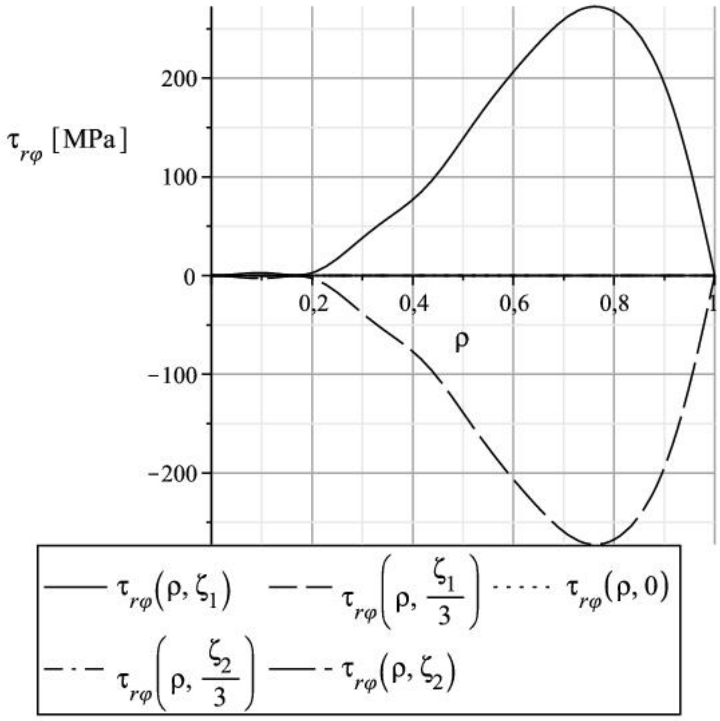

Figure 4. The graph of $\tau_{r\varphi}(\rho,\zeta_i)$.

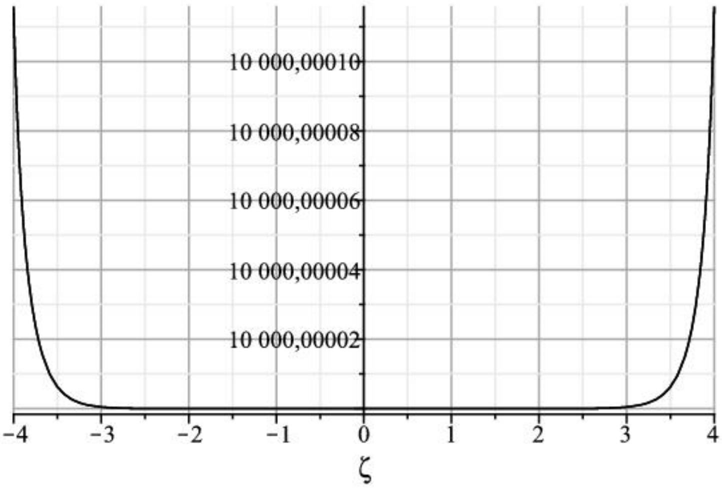

Figure 5. The plot of $t = t(\zeta)$ as a function of ζ.

4. CONCLUSIONS

To describe the torsional deformation of compound circular bar with solid cross section an analytical solution is developed which is based on the Michell-Föppl theory of torsion of body of rotation. The numerical results of the paper can be used as benchmark solution to check the accuracy of different approximate method used in structural mechanics.

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REFERENCES

[1] ARUTYUNYAN, N. Kh., ABRAMYAN, B. L.: **Torsion of Elastic Bodies.** (in Russian), Fizmatgiz, Moscow, 1963.
[2] SOKOLNIKOFF, I. S.: **Mathematical Theory of Elasticity.** McGraw-Hill, New York, 1970, 2nd Edition
[3] LEKHNITSKII, S. G.: **Theory of Elasticity of an Anisotropic Body.** MIR Publishers, Moscow, 1981.
[4] LEKHNITSKII, S. G.: **Theory of Anisotropic and Nonhomogeneous Bars.** (in Russian) Fizmatgiz, Moscow, 1971.
[5] RENTON, J. D.: **Elastic Beams and Frames.** Camford Books, Oxford, 1999.